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Generating primitive positive clones

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Abstract. Suppose $\mathcal{F}$ is a set of operations on a finite set A. Define PPC($\mathcal{F}$) to be the smallest primitive positive clone on A containing $\mathcal{F}$. For any finite algebra **A**, let PPC#(**A**) be the smallest number n for which PPC(Clo**A**) = PPC(Clo$_n$**A**). S. Burris and R. Willard [2] conjectured that PPC#(**A**) $\leq$ |**A**| when Clo**A** is a primitive positive clone and |**A**| $>$ 2. In this paper, we look at how large PPC#(**A**) can be when special conditions are placed on the finite algebra **A**. We show that PPC#(**A**) $\leq |A|$ holds when the variety generated by **A** is congruence distributive, Abelian, or decidable. We also show that PPC#(**A**) $\leq |A| + 2$ if **A** generates a congruence permutable variety and every subalgebra of **A** is the product of a congruence neutral algebra and an Abelian algebra. Furthermore, we give an example in which PPC#(**A**) $\geq (|\mathbf{A}| - 1)^2$ so that these results are not vacuous.

1. Introduction

A **primitive positive formula** is a first order formula of the form $\exists \wedge$ (atomic). A clone $\mathcal{C}$ on a set A is a **primitive positive clone** if every operation on A defined from operations in $\mathcal{C}$ using a primitive positive formula is already in $\mathcal{C}$. S. Burris and R. Willard proved in [2] that there are finitely many primitive positive clones on any finite set. They also claim that every primitive positive clone on a finite set A is generated using primitive positive definitions from its members of rank $|A|^{|A|}$. They conjecture that every such primitive positive clone is actually generated from its members of rank $|A|$ if A has more than two elements.

Suppose that $\mathcal{F}$ is a set of operations on a finite set A. Define PPC($\mathcal{F}$) to be the smallest primitive positive clone on A containing $\mathcal{F}$. For any finite algebra **A**, let

$$\mathrm{PPC\#}(\mathbf{A}) = \min\{n : \mathrm{PPC}(\mathrm{Clo}\mathbf{A}) = \mathrm{PPC}(\mathrm{Clo}_n\mathbf{A})\}.$$

The Burris-Willard conjecture is that PPC#(**A**) $\leq$ |**A**| when Clo**A** is a primitive positive clone and |**A**| $>$ 2. In this paper we look at how large PPC#(**A**) can be when special conditions are placed on the finite algebra **A**.

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It follows from Burris [1] that PPC#($\mathbf{A}$) $\leq |A|$ when $\mathbf{A}$ is a hereditarily subdirectly irreducible algebra in a congruence distributive variety or an algebra which has a linearly ordered lattice as a reduct. We show that this inequality also holds when the variety generated by $\mathbf{A}$ is congruence distributive, Abelian, or decidable. We also show that PPC#($\mathbf{A}$) $\leq |A| + 2$ if $\mathbf{A}$ generates a congruence permutable variety and every subalgebra of $\mathbf{A}$ is the product of a congruence neutral algebra and an Abelian algebra. Furthermore, we give an example in which PPC#($\mathbf{A}$) $\geq (|\mathbf{A}| - 1)^2$ so that these results are not vacuous.

2. Preliminaries

If $\mathbf{A}$ and $\mathbf{B}$ are similar algebras, let $\mathrm{Hom}(\mathbf{A}, \mathbf{B})$ be the set of all homomorphisms from $\mathbf{A}$ to $\mathbf{B}$. For any algebra $\mathbf{A}$, let $\mathbf{Z}(\mathbf{A})$ denote the clone $\cup_{n=1}^{\infty} \mathrm{Hom}(\mathbf{A}^n, \mathbf{A})$. This is called the **centralizer clone** of $\mathbf{A}$, and the operations in $\mathbf{Z}(\mathbf{A})$ are said to **centralize** the operations of $\mathbf{A}$. For any finite set A, the primitive positive clones on A correspond to the centralizer clones on A. In fact, $\mathrm{PPC}(\mathrm{Clo}\mathbf{A}) = \mathbf{Z}(\mathbf{Z}(\mathbf{A}))$.

Suppose f is an n-ary operation on a set A and t is any operation on A. We say that f **respects** or **preserves** t if f is a homomorphism from $\langle A, t\rangle^n$ to $\langle A, t\rangle$.

The following theorem and lemma will be essential throughout the paper. Theorem 2.1 is due to Kuznecov (see [3] and [16]). The Proof of Lemma 2.2 was inspired by the Proof of Theorem 2 of [2].

THEOREM 2.1. ([3] and [16]) *Suppose $\mathcal{F}_1$ and $\mathcal{F}_2$ are sets of operations on a finite set A.*

$$PPC(\mathcal{F}_1) = PPC(\mathcal{F}_2) \textit{ if and only if } \mathbf{Z}(\langle A, \mathcal{F}_1\rangle) = \mathbf{Z}(\langle A, \mathcal{F}_2\rangle).$$

LEMMA 2.2. (see [2]) *Suppose that $\mathbf{A}$ is a finite algebra with k members, n is a positive integer, and $f : A^n \to A$ is any n-ary operation on A. Then $f \in \mathrm{Hom}(\mathbf{A}^n, \mathbf{A})$ if and only if $\ker f \in \mathrm{Con}\mathbf{A}^n$ and f respects the k-ary term operations of $\mathbf{A}$.*

Proof. If $f \in \mathrm{Hom}(\mathbf{A}^n, \mathbf{A})$, then $\ker f \in \mathbf{A}^n$ and f respects the k-ary term operations of $\mathbf{A}$. Suppose then that $f \notin \mathrm{Hom}(\mathbf{A}^n, \mathbf{A})$ but that $\ker f \in \mathrm{Con}\mathbf{A}^n$. We show that f fails to respect some k-ary term operation of $\mathbf{A}$. Since $\ker f \in \mathrm{Con}\mathbf{A}^n$, the algebra $\mathbf{A}^n$ induces the natural algebraic structure on $\mathbf{A}^n/\ker f$ and the canonical surjection $\pi : \mathbf{A}^n \to \mathbf{A}^n/\ker f$ is a homomorphism. Let $\bar{f} : \mathbf{A}^n/\ker f \to \mathbf{A}$ be the unique map for which $f = \bar{f} \circ \pi$. Since f is not a homomorphism, the function $\bar{f}$ is not a homomorphism. This means there is a term $t(x_1, \ldots, x_m)$ in the language of $\mathbf{A}$ and elements $a_1, \ldots, a_m$ of $\mathbf{A}^n/\ker f$ so that

$$\bar{f}(t^{A^n/\ker f}(a_1, \ldots, a_m)) \neq t^A(\bar{f}(a_1), \ldots, \bar{f}(a_m)).$$

Since $\ker \pi = \ker f$, the function $\bar{f}$ is injective. Therefore, by identifying variables we can assume that $m \leq k$. On the other hand, by repeating variables if necessary, we can assume that $k \leq m$. Thus we can assume that $m = k$. For $i = 1, \ldots, m = k$, select $b_i \in \mathbf{A}^n$ so that $\pi(b_i) = a_i$. The inequality above along with the choice of $\bar{f}$ gives

$$f(t^{A^n}(b_1, \ldots, b_k)) \neq t^A(f(b_1), \ldots, f(b_k)).$$

Thus f fails to respect the k-ary term operation t^A of $\mathbf{A}$. □

We will also need this fact (which was brought to our attention by R. McKenzie):

LEMMA 2.3. *The congruences of a k-element algebra are determined by its k-ary term operations.*

Proof. Suppose $\mathbf{A}$ is a k element algebra. We show that the principal congruences of $\mathbf{A}$ are determined by $\mathrm{Clo}_k\mathbf{A}$. Let $a, b \in A$ and let θ be the principal congruence generated by $\langle a, b\rangle$. Suppose p is any unary polynomial of $\mathbf{A}$. We use k-ary term operations to witness that $\langle p(a), p(b)\rangle \in \theta$. This will suffice. There is some term operation t of $\mathbf{A}$ so that $p(x) = t(x, \bar{c})$ for some tuple $\bar{c}$ of elements of A. We can assume that $\bar{c}$ is a k-tuple and that the first two members of the tuple are a and b. Hence $p(x) = t(x, a, b, \bar{d})$ for an appropriate $\bar{d}$. Let $g(x) = t(a, a, x, \bar{d})$ and $h(x) = t(x, a, x, \bar{d})$, so that g and h are unary polynomials derived from k-ary term operations. Since $\langle a, b\rangle \in \theta$, $h(\langle a, b\rangle) = \langle h(a), p(b)\rangle \in \theta$ and $g(\langle b, a\rangle) = \langle p(a), g(a)\rangle \in \theta$. But, $h(a) = g(a)$, so

$$\langle p(a), p(b)\rangle = \langle p(a), g(a)\rangle \circ \langle h(a), p(b)\rangle \in \theta.$$

□

Finally, we note that if $n \leq k$ are integers and $\mathbf{A}$ is any algebra, then $\mathrm{Clo}_k\mathbf{A}$ completely determines $\mathrm{Clo}_n\mathbf{A}$. In particular, if $\mathbf{A}'$ is another algebra with the same universe as $\mathbf{A}$ and $\mathrm{Clo}_k\mathbf{A} = \mathrm{Clo}_k\mathbf{A}'$, then $\mathrm{Clo}_n\mathbf{A} = \mathrm{Clo}_n\mathbf{A}'$.

3. A large example

We now turn our attention to determining how large PPC#($\mathbf{A}$) can be. We show that in general PPC#($\mathbf{A}$) $< (|\mathbf{A}| - 1)^2$ does not always hold. This example is an extension of an example shown to us by R. McKenzie.

THEOREM 3.1. *For any integer $k > 2$ and any k element set A, there is a $(k-1)^2$-ary operation T and a binary operation $\wedge$ on A so that $\wedge$ is not in the centralizer of $\mathbf{A} = \langle A, T\rangle$, but $\wedge$ is in the centralizer of $\langle A, \mathrm{Clo}_{(k-1)^2-1}\mathbf{A}\rangle$.*

Proof. Let $A = \{0, 1, \ldots, k-1\}$, and let $\wedge$ be the meet associated with the flat semilattice on A with least element 0. Define the following $(k-1)^2$-tuples of elements of A:

$$f = \langle 1, 2, \ldots, k-1, 1, 2, \ldots, k-1, \ldots, 1, 2 \ldots, k-1\rangle$$

$$g = \langle 1, 1, \ldots, 1, 2, 2, \ldots, 2 \ldots, k-1, k-1, \ldots, k-1 \rangle.$$

(There are $k-1$ repetitions of $1, 2, \ldots, k-1$ in f, and each constant subsequence of g is of length $k-1$.) Let T be given by:

$$T(x_1, \ldots, x_{(k-1)^2}) = \begin{cases} 1 & \langle x_1, \ldots, x_{(k-1)^2} \rangle \in \{f, g\} \\ 0 & \text{else} \end{cases}$$

The operation $\wedge$ clearly does not centralize T for

$$T(f) \wedge T(g) = 1 \neq 0 = T(f \wedge g).$$

To make things simpler, let $n = (k-1)^2$. Suppose that q is an $(n-1)$-ary term operation derived from T. We show by induction on the complexity of q that $q(x_1, \ldots, x_{n-1}) = q(y_1, \ldots, y_{n-1}) \neq 0$ implies that the $\langle x_1, \ldots, x_{n-1} \rangle$ and $\langle y_1, \ldots, y_{n-1} \rangle$ coincide in the coordinates essential to q. It will then follow easily that $\wedge$ centralizes q. Suppose first that $q = T(p_1, \ldots, p_n)$ where $p_1, \ldots, p_n$ are projections. Assume that $\bar{x} = \langle x_1, \ldots, x_{n-1} \rangle \in A^{n-1}$, $\bar{y} = \langle y_1, \ldots, y_{n-1} \rangle \in A^{n-1}$, and

$$q(x_1, \ldots, x_{n-1}) = q(y_1, \ldots, y_{n-1}) \neq 0.$$

It must be that $q(\bar{x}) = q(\bar{y}) = 1$ since the image of T is $\{0, 1\}$. Because $q(\bar{x}) = q(\bar{y}) = 1$, it is necessary that

$$\langle p_1(\bar{x}), \ldots, p_n(\bar{x}) \rangle, \langle p_1(\bar{y}), \ldots, p_n(\bar{y}) \rangle \in \{f, g\}.$$

Since we only have $n-1$ projections to work with, there must be $i \neq j$ so that $p_i = p_j$. Either $f_i \neq f_j$ or $g_i \neq g_j$ (if we treat f and g as functions their kernels intersect to the identity). Without loss of generality, assume $f_i \neq f_j$. Then it must be that

$$\langle p_1(\bar{x}), \ldots, p_n(\bar{x}) \rangle = g = \langle p_1(\bar{y}), \ldots, p_n(\bar{y}) \rangle.$$

If q depends on the l^{th} coordinate, then for some i the operation p_i is the projection to the l^{th} coordinate. Therefore:

$$x_l = p_i(\bar{x}) = g_i = p_i(\bar{y}) = y_l.$$

Thus we see that if q relies on its l^{th} coordinate, then $x_l = y_l$. This establishes the base of our induction.

Now suppose that $q = T(q_1, \ldots, q_n)$ where each q_i is $(n-1)$-ary. Assume also that for each q_i and each $\bar{x} = \langle x_1, \ldots, x_{n-1} \rangle \in A^{n-1}$ and $\bar{y} = \langle y_1, \ldots, y_{n-1} \rangle \in A^{n-1}$ if $q_i(\bar{x}) = q_i(\bar{y}) \neq 0$ then $\bar{x}$ and $\bar{y}$ agree in the coordinates essential to q_i. Suppose that $q(\bar{x}) = q(\bar{y}) \neq 0$. Again, it is necessary that $q(\bar{x}) = q(\bar{y}) = 1$ and that

$$\langle q_1(\bar{x}), \ldots, q_n(\bar{x}) \rangle, \langle q_1(\bar{y}), \ldots, q_n(\bar{y}) \rangle \in \{f, g\}.$$

If $\langle q_1(\bar{x}), \ldots, q_n(\bar{x})\rangle = \langle q_1(\bar{y}), \ldots, q_n(\bar{y})\rangle$, then it follows by induction that the variables in $\bar{x}$ and $\bar{y}$ essential to q are equal. We show that this is the only possibility.

Assume by way of contradiction that

$$\langle q_1(\bar{x}), \ldots, q_n(\bar{x})\rangle = f \text{ and } \langle q_1(\bar{y}), \ldots, q_n(\bar{y})\rangle = g.$$

For $i > 1$ either $q_i(\bar{x})$ or $q_i(\bar{y})$ fails to be 0 or 1. This means that q_i must be a projection for $i > 1$ (any term operation more complicated than a projection can output only 0 or 1). If for some $i \neq j > 1$ we had $q_i = q_j$, then our previous observation about the kernels of f and g will again show that $\langle q_1(\bar{x}), \ldots, q_n(\bar{x})\rangle$ and $\langle q_1(\bar{y}), \ldots, q_n(\bar{y})\rangle$ must both be f or both be g - contrary to our assumption. Assume then that for $i > 1$ the q_i are distinct projections. If q_1 is a projection, we are in the base case of our induction. Suppose then that q_1 is not a projection. Since q_1 is not a projection and $q_1(\bar{x}) = q_1(\bar{y}) = 1$, q_1 must depend on at least $k-1$ variables. We claim that q_1 must depend on some coordinate j for which $x_j \neq y_j$. From the way in which f and g were defined, we know there are exactly $k-2$ indices $i > 1$ where $q_i(\bar{x}) = f_i = g_i = q_i(\bar{y})$. Since $q_2, \ldots, q_n$ account for all $n-1$ of the $(n-1)$-ary projections, this means that there are exactly $k-2$ coordinates i where $x_i = y_i$. Since q_1 depends on at least $k-1$ coordinates, q_1 depends on one of the coordinates j where $x_j \neq y_j$. However, we know $q_1(\bar{x}) = q_1(\bar{y}) = 1$, so by induction $\bar{x}$ and $\bar{y}$ must agree in the coordinates essential to q_1 - and hence at the j^{th} coordinate. This contradiction refutes the assumption that $\langle q_1(\bar{x}), \ldots, q_n(\bar{x})\rangle \neq \langle q_1(\bar{y}), \ldots, q_n(\bar{y})\rangle$ and completes the argument that if $q(\bar{x}) = q(\bar{y}) \neq 0$, then $\bar{x}$ and $\bar{y}$ agree in the coordinates essential to q.

By induction then, for any $(n-1)$-ary term operation q derived from T and for any $a \neq 0$ there is at most one way to chose the variables in $\bar{x}$ essential to q so that $q(\bar{x}) = a$. From this it is easy to prove that $\wedge$ centralizes q. □

In this theorem, $\mathbf{Z}(\mathbf{A})$ is not determined by the operations of $\mathbf{A}$ with rank less than $(|A|-1)^2$. Hence PPC#($\mathbf{A}$) is at least $(|A|-1)^2$. Thus:

COROLLARY 3.2. *Suppose A is a finite set with at least three elements. There is an algebra $\mathbf{A}$ on A for which PPC#($\mathbf{A}$) $\geq (|A|-1)^2$.*

4. When PPC#(A) is small

We now turn our attention to situations in which PPC#($\mathbf{A}$) can be shown to be "small". We begin by looking at the primitive positive clone generated by the term operations of an algebra in a congruence distributive variety.

THEOREM 4.1. *Suppose that $\mathbf{A}$ is a finite algebra with at least three elements that generates a congruence distributive variety.*

$$PPC\#(\mathbf{A}) \leq |A|.$$

Proof. Suppose that $|A| = k$, and let $\mathbf{A}' = \langle A, \mathrm{Clo}_k(\mathbf{A})\rangle$. Since $\mathbf{A}$ and $\mathbf{A}'$ have the same k-ary term operations, it follows from Lemma 2.3 that $\mathbf{A}$ and $\mathbf{A}'$ have precisely the same congruences. Since $k \geq 3$ and $\mathbf{A}$ and $\mathbf{A}'$ have the same k-ary term operations, $\mathbf{A}'$ shares the ternary Jónnson terms of $\mathbf{A}$ which witness to congruence distributivity. Hence the variety generated by $\mathbf{A}'$ is congruence distributive. We show that $\mathbf{Z}(\mathbf{A}) = \mathbf{Z}(\mathbf{A}')$. Suppose that n is a positive integer and f is a homomorphism from $(\mathbf{A}')^n$ to $\mathbf{A}'$. We will show that f is a homomorphism from $\mathbf{A}^n$ to $\mathbf{A}$. We first show that $\ker f \in \mathrm{Con}\mathbf{A}^n$. For $i = 1, \ldots, n$ let π_i be the projection of A^n to the i^{th} coordinate and let $\eta_i = \ker \pi_i$. By the Correspondence Theorem, $\ker f \vee \eta_i = \pi_i^{-1}(\pi_i(\ker f \vee \eta_i))$ for each i. Since $\pi_i(\ker f \vee \eta_i) \in \mathrm{Con}\mathbf{A}' = \mathrm{Con}\mathbf{A}$, this means that $\ker f \vee \eta_i$ is a congruence on $\mathbf{A}^n$. By the congruence distributivity of $\mathcal{V}(\mathbf{A}')$:

$$\begin{aligned} \ker f &= \ker f \vee 0_{A^n} \\ &= \ker f \vee \left(\bigcap_{i=1}^{n} \eta_i\right) \\ &= \bigcap_{i=1}^{n} (\ker f \vee \eta_i)\,. \end{aligned}$$

Thus $\ker f$ is the intersection of members of $\mathrm{Con}(\mathbf{A}^n)$, so $\ker f$ is also a congruence on $\mathbf{A}^n$. Since $\mathbf{A}$ and $\mathbf{A}'$ have the same k-ary term operations, we already know that f preserves the k-ary term operations of $\mathbf{A}$. It follows from Lemma 2.2 that $f : \mathbf{A}^n \to \mathbf{A}$ is a homomorphism. Thus $\mathbf{Z}(\mathbf{A}') \subseteq \mathbf{Z}(\mathbf{A})$. Since the reverse inclusion is trivial, $\mathbf{A}$ and $\mathbf{A}'$ have the same centralizer clone and thus determine the same primitive positive clone by Theorem 2.1. □

We next show that in almost all cases if $\mathbf{A}$ is a finite algebra generating an Abelian variety then $\mathrm{PPC\#}(\mathbf{A}) \leq |A|$. We first need to note some results and definitions from the literature. An algebra $\mathbf{A}$ is called **Hamiltonian** if and only if every subalgebra of $\mathbf{A}$ is an equivalence class of a congruence on $\mathbf{A}$. An algebra $\mathbf{A}$ is said to have the **congruence extension property** if and only if for every subalgebra $\mathbf{B}$ of $\mathbf{A}$ and for any congruence θ on $\mathbf{B}$ there is a congruence ψ on $\mathbf{A}$ so that $\psi \cap B^2 = \theta$.

Suppose τ is the type of a universal algebra. A **principal congruence formula** of type τ is a first order formula $\pi(x, y, u, v)$ of the form

$$\exists \bar{w} \left(x \approx p_1(z_1, \bar{w}) \wedge \left[\bigwedge_{i=1}^{n-1} p_i(z_i', \bar{w}) \approx p_{i+1}(z_{i+1}, \bar{w}) \right] \wedge p_n(z_n', \bar{w}) \approx y \right)$$

where $\{z_i, z_i'\} = \{u, v\}$ and p_i is a τ-term for each i. It follows from the familiar theorem of Maltsev that for any algebra $\mathbf{A}$ of type τ and any $a, b, c, d \in \mathbf{A}$ it is the case that $\langle a, b\rangle \in \mathrm{Cg}_{\mathbf{A}}(\langle c, d\rangle)$ (the principal congruence on $\mathbf{A}$ generated by $\langle c, d\rangle$) if and only if $\mathbf{A} \models \pi(a, b, c, d)$ for some principal congruence formula π of type τ.

The congruence extension property gives a means of limiting the rank of terms occurring in the principal congruence formulas necessary to define all principal congruences. This follows from the following theorem of A. Day:

THEOREM 4.2. [4] *For any algebra* $\mathbf{A}$, *the following are equivalent:*

1. $\mathbf{A}$ *satisfies the congruence extension property.*
2. *For all* $a, b, c, d \in A$, *if* $\mathbf{B} = \mathrm{Sg}_{\mathbf{A}}(\{a, b, c, d\})$ *then* $\langle a, b\rangle \in \mathrm{Cg}_{\mathbf{A}}(\langle c, d\rangle)$ *if and only if* $\langle a, b\rangle \in \mathrm{Cg}_{\mathbf{B}}(\langle c, d\rangle)$.

Thus to determine if $\langle a, b\rangle \in \mathrm{Cg}_{\mathbf{A}}(c, d)$, it is enough to know that $\mathbf{B} \models \pi(a, b, c, d)$ for some principal congruence formula π in the type of $\mathbf{A}$. Since $\mathbf{B}$ is generated by $\{a, b, c, d\}$, it is enough to consider principal congruence formulas which employ only 5-ary terms.

E. Kiss has shown:

THEOREM 4.3. [9] *Every Hamiltonian variety satisfies the congruence extension property.*

E. Kiss and M. Valeriote give us:

THEOREM 4.4. [8] *Suppose* $\mathcal{V}$ *is a locally finite variety.* $\mathcal{V}$ *is Hamiltonian if and only if* $\mathcal{V}$ *is Abelian.*

We are almost ready for our theorem. We need one more background result. We will need that the variety generated by a reduct of an algebra which generates a Hamiltonian variety is also Hamiltonian. The next theorem insures this as long as the reduct contains all ternary term operations of $\mathbf{A}$:

THEOREM 4.5. [11] *A variety* $\mathcal{V}$ *is Hamiltonian if and only if for every* $(n+1)$-*ary term* t *of* $\mathcal{V}$ *there is a ternary term* s *of* $\mathcal{V}$ *so that*

$$\mathcal{V} \models s(x, y, t(x, z_1, \dots, z_n)) \approx t(y, z_1, \dots, z_n).$$

THEOREM 4.6. *Suppose* $\mathbf{A}$ *is a finite algebra with k elements. If* $\mathbf{A}$ *generates an Abelian variety, then PPC#*$(\mathbf{A}) \leq$ *max*$(5, |A|)$.

Proof. Let $n = \max(5, |A|)$. Suppose $\mathbf{A}$ generates an Abelian variety. By Theorem 4.4, every algebra in the variety generated by $\mathbf{A}$ is Hamiltonian. Let $\mathbf{A}' = \langle A, \mathrm{Clo}_n\mathbf{A}\rangle$. By Theorem 4.5 every algebra in the variety generated by $\mathbf{A}'$ is also Hamiltonian. By Theorem 4.3 both of these varieties also satisfy the congruence extension property. By Theorem 4.2, the principal congruence formulas derived from 5-ary terms of each of these varieties are adequate to define all principal congruences on all algebras in the varieties.

Since $\mathbf{A}$ and $\mathbf{A}'$ have the same 5-ary term operations, we can conclude that $\mathbf{A}^m$ and $(\mathbf{A}')^m$ have the same 5-ary term operations and hence the same principal congruences for any m. Therefore, they also have the same congruences. Suppose that $f : (\mathbf{A}')^m \to \mathbf{A}'$ is a homomorphism. Then $\ker f \in \mathrm{Con}\mathbf{A}^m$. Since $\mathbf{A}$ and $\mathbf{A}'$ share the same n-ary term operations, f respects these. Thus f is a homomorphism from $\mathbf{A}^m$ to $\mathbf{A}$ by Lemma 2.2. Since every homomorphism from $\mathbf{A}^m$ to $\mathbf{A}$ is also a homomorphism from $(\mathbf{A}')^m$ to $\mathbf{A}'$, we see that $\mathbf{A}$ and $\mathbf{A}'$ have the same centralizer clone. The result follows from Theorem 2.1. □

An algebra is **affine** if it has a Maltsev term and is Abelian. It is a simple matter to show that the clone of an affine algebra is generated by its term operations with rank at most three. Since the ternary term operations of an algebra completely determine the unary and binary term operations, the clone of an affine algebra is generated by its ternary term operations. Thus the centralizer clone of an affine algebra is determined by its term operations of rank three. The next theorem is now immediate.

THEOREM 4.7. *If* $\mathbf{A}$ *is a finite affine algebra, then* $PPC\#(\mathbf{A}) \leq 3$.

We say that varieties $\mathcal{V}_1, \dots, \mathcal{V}_n$ in the same type are **independent** if and only if there is an n-ary term t in the language of the varieties so that

$$\mathcal{V}_i \models t(x_1, \dots, x_n) \approx x_i$$

for $i = 1, \dots, n$. If $\mathcal{V}_1, \dots, \mathcal{V}_n$ are independent, then every algebra in the variety $\mathcal{V}$ generated by $\bigcup_{i=1}^n \mathcal{V}_i$ is isomorphic to a product $\prod_{i=1}^n \mathbf{A}_i$ where each $\mathbf{A}_i \in \mathcal{V}_i$. In this case we write $\mathcal{V} = \mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_n$. Whenever $\mathbf{A} = \mathbf{A}_1 \times \cdots \times \mathbf{A}_n$ in $\mathcal{V}$ in this manner, we write $\mathbf{A} = \mathbf{A}_1 \otimes \cdots \otimes \mathbf{A}_n$. If $\mathcal{V}_1$, $\mathcal{V}_2$, and $\mathcal{V}_3$ are independent, then $\mathcal{V}_1$ and $\mathcal{V}_2$ are independent. Also, $\mathcal{V}_1 \otimes \mathcal{V}_2$ and $\mathcal{V}_3$ are independent and $\mathcal{V}_1 \otimes \mathcal{V}_2 \otimes \mathcal{V}_3 = (\mathcal{V}_1 \otimes \mathcal{V}_2) \otimes \mathcal{V}_3$. The operator PPC# interacts nicely with $\otimes$:

LEMMA 4.8. *Suppose* $\mathbf{A} = \mathbf{A}_1 \otimes \cdots \otimes \mathbf{A}_n$. *If for some integer* m $PPC\#(\mathbf{A}_i) \leq m$ *for* $i = 1, \dots, n$, *then* $PPC\#(\mathbf{A}) \leq m$.

Proof. It is not too difficult to see that a function $f : A^k \to A$ is a homomorphism from $\mathbf{A}^k$ to $\mathbf{A}$ if and only if there are homomorphisms $f_i : \mathbf{A}_i^k \to \mathbf{A}_i$ for $i = 1, \dots, n$ so that

$$\begin{aligned} &f(\langle x_1^1, x_2^1, \dots, x_n^1\rangle, \langle x_1^2, x_2^2, \dots, x_n^2\rangle, \dots, \langle x_1^k, x_2^k, \dots, x_n^k\rangle) \\ &\quad = \langle f_1(x_1^1, x_1^2, \dots, x_1^k), f_2(x_2^1, x_2^2, \dots, x_2^k), \dots, f_n(x_n^1, x_n^2, \dots, x_n^k)\rangle. \end{aligned}$$

Thus $\mathbf{Z}(\mathbf{A}) = \mathbf{Z}(\mathbf{A}_1) \times \cdots \times \mathbf{Z}(\mathbf{A}_n)$. For each i, $\mathrm{PPC\#}(\mathbf{A}_i) \leq m$ implies that $\mathbf{Z}(\mathbf{A}_i)$ is determined by $\mathrm{Clo}_m\mathbf{A}_i$. Thus $\mathbf{Z}(\mathbf{A})$ is determined by the collection $\{\mathrm{Clo}_m\mathbf{A}_i\}_{i=1}^n$. This collection is determined by $\mathrm{Clo}_m\mathbf{A}$. By Theorem 2.1 it follows that $\mathrm{PPC\#}(\mathbf{A})$ is at most m. □

Applying this lemma to Theorems 4.1 and 4.6, we get the following corollary:

COROLLARY 4.9. *Suppose* $\mathbf{A}$ *is a finite algebra and* $\mathbf{A} = \mathbf{A}_1 \otimes \mathbf{A}_2$ *where* $\mathbf{A}_1$ *is in an Abelian variety,* $\mathbf{A}_2$ *is in a congruence distributive variety, and* $|\mathbf{A}_2| \geq 3$. *PPC#*$(\mathbf{A}) \leq$ *max*$(5, |A|)$.

We say that a variety $\mathcal{V}$ is **decidable** if and only if there is an algorithm which, given any sentence in the language of $\mathcal{V}$, decides if that sentence holds in every algebra in $\mathcal{V}$. In [14], R. McKenzie and M. Valeriote give the following characterization of decidable locally finite varieties:

THEOREM 4.10. [14] *Let* $\mathcal{V}$ *be any decidable locally finite variety. There exists a strongly Abelian variety* $\mathcal{V}_1$*, an affine variety* $\mathcal{V}_2$*, and a discriminator variety* $\mathcal{V}_3$ *so that* $\mathcal{V} = \mathcal{V}_1 \otimes \mathcal{V}_2 \otimes \mathcal{V}_3$.

All we need to know about these varieties is that $\mathcal{V}_1 \otimes \mathcal{V}_2$ will be Abelian since $\mathcal{V}_1$ and $\mathcal{V}_2$ are, and that $\mathcal{V}_3$ is congruence distributive (as a discriminator variety). Thus if an algebra $\mathbf{A}$ is contained in a decidable variety, that algebra is always isomorphic to

$$\mathbf{A}_1 \otimes \mathbf{A}_2 \otimes \mathbf{A}_3 \cong (\mathbf{A}_1 \otimes \mathbf{A}_2) \otimes \mathbf{A}_3$$

where $(\mathbf{A}_1 \otimes \mathbf{A}_2)$ is in an Abelian variety and $\mathbf{A}_3$ is in a congruence distributive variety. Hence:

THEOREM 4.11. *Suppose* $\mathbf{A}$ *is a finite algebra in a decidable variety. PPC#*$(\mathbf{A}) \leq$ *max*$(5, |A|)$.

In order to discuss the next class of algebras we would like to address, we need some definitions. Suppose α, β, and γ are congruences on an algebra $\mathbf{A}$. We say that α **centralizes** β modulo γ and write $C(\alpha, \beta; \gamma)$ if and only if for all term operations t of $\mathbf{A}$ and for all elements $a, b, x_1, \ldots, x_n, y_1, \ldots, y_n$ of $\mathbf{A}$ if $a\alpha b$ and $x_i \beta y_i$ for $i = 1, \ldots, n$, then

$$\begin{array}{c} t(a, x_1, \ldots, x_n) \;\; \gamma \;\; t(a, y_1, \ldots, y_n) \\ \Updownarrow \\ t(b, x_1, \ldots, x_n) \;\; \gamma \;\; t(b, y_1, \ldots, y_n) \end{array}$$

We will write $C^2(\alpha, \beta; \gamma)$ if the above equivalence holds when it is also assumed that $x_i = y_i$ for $i = 2, \ldots, n$. The **commutator** of α and β - denoted by $[\alpha, \beta]$ - is the smallest congruence δ on $\mathbf{A}$ satisfying $C(\alpha, \beta; \delta)$. If $[\alpha, \beta] = \alpha \cap \beta$ for all α and β in Con$\mathbf{A}$, we say $\mathbf{A}$ is **congruence neutral** or simply **neutral**.

The commutator is particularly well-behaved for algebras in congruence modular varieties. For a thorough discussion of the commutator in congruence modular varieties, see [6]. If $\{\mathbf{A}_i\}_{i=1}^n$ are congruence neutral in a congruence modular variety, then all finite subdirect products of $\{\mathbf{A}_i\}_{i=1}^n$ have distributive congruence lattices. In a congruence modular variety, $\mathbf{A}$ is Abelian if and only if $[1_A, 1_A] = 0_A$. If $\psi : \mathbf{A} \to \mathbf{B}$ is a surjective homomorphism in a congruence modular variety and α and β are congruences above ker ψ, then $\psi^{-1}([\psi(\alpha), \psi(\beta)]) = [\alpha, \beta] \vee \ker \psi$. From this, it follows that if $\mathbf{A}$ is Abelian (or neutral), then so is $\mathbf{B}$. Recall that a congruence permuting variety is also congruence modular. The notion of centralization can be captured with the C^2 relation in the presence of a Maltsev term:

LEMMA 4.12. *If an algebra* $\mathbf{A}$ *has a Maltsev term, then* $C(\alpha, \beta; \gamma)$ *is equivalent to* $C^2(\alpha, \beta; \gamma)$ *for all congruences* α, β, *and* γ *of* $\mathbf{A}$.

This lemma follows from Lemma 2.8 of [10]. We are now ready to define the next class of algebras with which we will be working. An algebra $\mathbf{A}$ is **hereditarily (neutral × Abelian)** if and only if every subalgebra of $\mathbf{A}$ is the product of a neutral algebra and an Abelian algebra. We will show here that the term operations of a hereditarily (neutral × Abelian) algebra $\mathbf{A}$ with a Maltsev term are generated by $\mathrm{Clo}_{|A|+2}\mathbf{A}$. This will immediately give us a result concerning PPC#($\mathbf{A}$). We first need a structure theory which will allow us to discuss subdirect products of congruence neutral algebras with Maltsev operations. We will extend a familiar lemma of I. Fleischer. We will be using the following rather technical notation in the next few pages. For a subdirect product (of sets) $B \subseteq \prod_{i \in I} A_i$, define π_i^B to be the projection of B to the i^{th} coordinate and $\eta_i^B = \ker(\pi_i^B)$ for $i \in I$. For $i \neq j \in I$, let $\delta_{\{i,j\}}^B : B \to B/(\eta_i^B \vee \eta_j^B)$ be the canonical map, and let $\alpha_{i,j}^B : A_i \to B/(\eta_i^B \vee \eta_j^B)$ be the unique function defined by $\delta_{\{i,j\}}^B(x) = \alpha_{i,j}^B \pi_i^B(x)$ (we know this exists since $\ker \pi_i^B \subseteq \ker \delta_{\{i,j\}}^B$).

LEMMA 4.13. (Fleischer's Lemma [5]) *Suppose that* $A \subseteq A_1 \times A_2$ *is a subdirect product (of sets) in which the projection kernels permute. Then*

$$A = \{\langle x_1, x_2\rangle \in A_1 \times A_2 : \alpha_{1,2}^A(x_1) = \alpha_{2,1}^A(x_2)\}.$$

We extend this lemma to subdirect products of any finite number of factors. We first need to know how the α's interact with projections.

LEMMA 4.14. *Suppose* $B \subseteq \prod_{i \in I} A_i$ *is a subdirect product (of sets). If* $J \subseteq I$ *and* $C \subseteq \prod_{i \in J} A_i$ *is the projection of* B *to the coordinates in* J, *then for all* $i, j \in J$ *and for all* $a \in \mathbf{A}_i$ *and* $b \in \mathbf{A}_j$

$$\alpha_{i,j}^B(a) = \alpha_{j,i}^B(b) \text{ if and only if } \alpha_{i,j}^C(a) = \alpha_{j,i}^C(b).$$

Proof. Denote the projection of B onto C by π. Select $a \in \mathbf{A}_i$ and $b \in \mathbf{A}_j$ so that $\alpha_{i,j}^B(a) = \alpha_{j,i}^B(b)$. There exist $\bar{x}, \bar{y} \in B$ so that $\pi_i^B(\bar{x}) = a$ and $\pi_j^B(\bar{y}) = b$. Since

$$\delta_{\{i,j\}}^B(\bar{x}) = \alpha_{i,j}^B \pi_i^B(\bar{x}) = \alpha_{i,j}^B(a) = \alpha_{j,i}^B(b) = \alpha_{j,i}^B \pi_j^B(\bar{y}) = \delta_{\{j,i\}}^B(\bar{y})$$

we have $\langle \bar{x}, \bar{y}\rangle \in \ker \delta^B_{\{i,j\}} = \eta^B_i \vee \eta^B_j$. Therefore,

$$\langle \pi(\bar{x}), \pi(\bar{y})\rangle \in \pi(\eta^B_i \vee \eta^B_j) = \pi(\eta^B_i) \vee \pi(\eta^B_j) = \eta^C_i \vee \eta^C_j = \ker \delta^C_{\{i,j\}}.$$

This means:

$$\begin{aligned} \alpha^C_{i,j}(a) &= \alpha^C_{i,j}(\pi^C_i(\pi(\bar{x}))) \\ &= \delta^C_{\{i,j\}}(\pi(\bar{x})) \\ &= \delta^C_{\{j,i\}}(\pi(\bar{y})) \\ &= \alpha^C_{j,i}(\pi^C_j(\pi(\bar{y}))) \\ &= \alpha^C_{j,i}(b) \end{aligned}$$

Next, suppose $a \in \mathbf{A}_i$, $b \in \mathbf{A}_j$ and $\alpha^C_{i,j}(a) = \alpha^C_{j,i}(b)$. There exist $\bar{u}, \bar{v} \in C$ so that $\pi^C_i(\bar{u}) = a$ and $\pi^C_j(\bar{v}) = b$. Also, there exist $\bar{x}, \bar{y} \in B$ so that $\pi(\bar{x}) = \bar{u}$ and $\pi(\bar{y}) = \bar{v}$. Just as in the reverse direction, our assumptions require $\langle \bar{u}, \bar{v}\rangle \in \eta^C_i \vee \eta^C_j$. Therefore,

$$\langle \pi(\bar{x}), \pi(\bar{y})\rangle = \langle \bar{u}, \bar{v}\rangle \in \eta^C_i \vee \eta^C_j = \pi(\eta^B_i) \vee \pi(\eta^B_j) = \pi(\eta^B_i \vee \eta^B_j).$$

Thus

$$\langle \bar{x}, \bar{y}\rangle \in \pi^{-1}(\pi(\eta^B_i \vee \eta^B_j)) = \eta^B_i \vee \eta^B_j = \ker \delta^B_{\{i,j\}}$$

(where the second to the last equality follows from the Correspondence Theorem). Now

$$\begin{aligned} \alpha^B_{i,j}(a) &= \alpha^B_{i,j}(\pi^B_i(\bar{x})) \\ &= \delta^B_{\{i,j\}}(\bar{x}) \\ &= \delta^B_{\{j,i\}}(\bar{y}) \\ &= \alpha^B_{j,i}(\pi^B_i(\bar{y})) \\ &= \alpha^B_{j,i}(b). \end{aligned}$$

□

We are now ready to prove our extension of Fleischer's Lemma:

LEMMA 4.15. (Extension of Fleischer's Lemma) *If $B \subseteq \prod_{i=1}^n A_i$ is a subdirect product (of sets) in which the projection kernels generate a distributive lattice of permuting equivalence relations, then*

$$B = \left\{ \langle x_1, \ldots, x_n\rangle \in \prod_{i=1}^n A_i : \alpha^B_{i,j}(x_i) = \alpha^B_{j,i}(x_j) \forall i \neq j \right\}.$$

Proof. Let B be such a subdirect product. We prove this by induction on n. The case of $n = 1$ is vacuous and the case of $n = 2$ is proven by Fleischer's Lemma. Assume then that

$n > 2$ and that the result holds for subdirect products of fewer than n of the A_i's. Let π be the projection of B onto the first $n-1$ coordinates and let

$$\eta = \ker(\pi) = \bigwedge_{i=1}^{n-1} \eta_i^B.$$

Also, let $C = \pi(B)$. From the induction hypothesis,

$$C = \left\{ \langle x_1, \dots, x_{n-1} \rangle \in \prod_{i=1}^{n-1} A_i : \alpha_{i,j}^C(x_i) = \alpha_{j,i}^C(x_j) \forall i \neq j < n \right\}.$$

By the previous lemma, this means

$$C = \left\{ \langle x_1, \dots, x_{n-1} \rangle \in \prod_{i=1}^{n-1} A_i : \alpha_{i,j}^B(x_i) = \alpha_{j,i}^B(x_j) \forall i \neq j < n \right\}.$$

Let B' be the set in the statement of the lemma which we hope to be B. Let $\bar{x} = \langle x_1, \dots, x_n \rangle \in B'$. First, note that $\langle x_1, \dots, x_{n-1} \rangle \in C$. Therefore, we can select $y \in A_n$ so that $\langle x_1, \dots, x_{n-1}, y \rangle \in B$. We can also select $\langle z_1, \dots, z_{n-1} \rangle \in \prod_{i=1}^{n-1} A_i$ so that $\langle z_1, \dots, z_{n-1}, x_n \rangle \in B$. By our assumption,

$$\delta_{\{i,n\}}^B(\langle x_1, \dots, x_{n-1}, y \rangle) = \alpha_{i,n}^B(x_i) = \alpha_{n,i}^B(x_n) = \delta_{\{i,n\}}^B(\langle z_1, \dots, z_{n-1}, x_n \rangle)$$

for all $i < n$ (the middle equality follows from $\bar{x} \in B'$). Thus

$$(\langle x_1, \dots, x_{n-1}, y \rangle, \langle z_1, \dots, z_{n-1}, x_n \rangle) \in \bigwedge_{i=1}^{n-1} \ker(\delta_{\{i,n\}}^B)$$

$$\begin{aligned} &= \textstyle\bigwedge_{i=1}^{n-1} (\eta_i^B \vee \eta_n^B) \\ &= \eta_n^B \vee (\textstyle\bigwedge_{i=1}^{n-1} \eta_i^B) \quad \text{(distributivity)} \\ &= \eta_n^B \vee \eta \\ &= \eta_n^B \circ \eta \quad \text{(permutability).} \end{aligned}$$

Thus there is some $\bar{x}' \in B$ so that

$$\langle z_1, \dots, z_{n-1}, x_n \rangle \eta_n^B \bar{x}' \eta \langle x_1, \dots, x_{n-1}, y \rangle.$$

By the definitions of these kernels, this simply means $\bar{x} = \bar{x}' \in B$. This gives $B' \subseteq B$. Since the reverse inclusion is clear, $B = B'$ as desired. □

The next lemma exposes the structure of hereditarily (neutral $\times$ Abelian) algebras in a Maltsev variety which we will exploit:

LEMMA 4.16. *Suppose* $\mathbf{A}$ *is a finite hereditarily (neutral* $\times$ *Abelian) algebra in a Maltsev variety.* $\operatorname{Clo}\mathbf{A}$ *is generated by* $\operatorname{Clo}_{|A|+2}\mathbf{A}$.

Proof. Write $\bar{\mathbf{A}}$ for $\mathbf{A}$, and let $k = |A|$. Let $\underline{\mathbf{A}}$ be the algebra on A whose basic operations are the $(k+2)$-ary term operations of $\bar{\mathbf{A}}$. We want to show that $\bar{\mathbf{A}}$ and $\underline{\mathbf{A}}$ have the same term operations. We need to make several observations:

Because the subuniverses of a k-element algebra are completely determined by its k-ary term operations:

CLAIM 1. *A subset* $C \subseteq A$ *is a subuniverse of* $\bar{\mathbf{A}}$ *if and only if it is a subuniverse of* $\underline{\mathbf{A}}$.

For any subuniverse C of $\bar{\mathbf{A}}$, let $\bar{\mathbf{C}}$ denote the subalgebra of $\bar{\mathbf{A}}$ with universe C, and let $\underline{\mathbf{C}}$ denote the corresponding subalgebra of $\underline{\mathbf{A}}$. Assume that $\bar{\mathbf{C}}$ is a subalgebra of $\bar{\mathbf{A}}$. Trivially, $\operatorname{Clo}_{k+2}\bar{\mathbf{C}} = \operatorname{Clo}_{k+2}\underline{\mathbf{C}}$. It follows from Lemma 2.3 that

CLAIM 2. $\bar{\mathbf{C}}$ *and* $\underline{\mathbf{C}}$ *have the same congruences.*

Let θ be any congruence on $\bar{\mathbf{C}}$. Then $\operatorname{Clo}_{k+2}\bar{\mathbf{C}}/\theta = \operatorname{Clo}_{k+2}\underline{\mathbf{C}}/\theta$, so:

CLAIM 3. $\bar{\mathbf{C}}/\theta$ *and* $\underline{\mathbf{C}}/\theta$ *have the same congruences and subuniverses.*

Suppose α, β and γ are congruences on $\bar{\mathbf{C}}/\theta$ (and hence also on $\underline{\mathbf{C}}/\theta$). To determine if $C^2(\alpha, \beta; \gamma)$ holds in $\bar{\mathbf{C}}/\theta$ (or $\underline{\mathbf{C}}/\theta$) it suffices to consider only $(k+2)$-ary operations - since $\bar{\mathbf{C}}/\theta$ (and $\underline{\mathbf{C}}/\theta$) contains at most k elements. Since $\bar{\mathbf{C}}/\theta$ and $\underline{\mathbf{C}}/\theta$ have the same $(k+2)$-ary term operations, it follows that $C^2(\alpha, \beta; \gamma)$ holds in $\bar{\mathbf{C}}/\theta$ if and only if it holds in $\underline{\mathbf{C}}/\theta$. From Lemma 4.12, $C(\alpha, \beta; \gamma)$ holds in $\bar{\mathbf{C}}/\theta$ if and only if it holds in $\underline{\mathbf{C}}/\theta$. It follows that

CLAIM 4. $\bar{\mathbf{C}}/\theta$ *is Abelian (or neutral) if and only if* $\underline{\mathbf{C}}/\theta$ *is.*

CLAIM 5. *Suppose that* $\bar{\mathbf{D}} \in \operatorname{Sub}\bar{\mathbf{A}}$ *and* $\sigma \in \operatorname{Con}\bar{\mathbf{D}}$. *Let* $f : C/\theta \to D/\sigma$ *be any set map. Then* f *is a homomorphism from* $\bar{\mathbf{C}}/\theta$ *to* $\bar{\mathbf{D}}/\sigma$ *if and only if it is a homomorphism from* $\underline{\mathbf{C}}/\theta$ *to* $\underline{\mathbf{D}}/\sigma$.

To see this, suppose that f is not a homomorphism from $\bar{\mathbf{C}}/\theta$ to $\bar{\mathbf{D}}/\sigma$. Then there is some term operation T of $\bar{\mathbf{A}}$ and elements $x_1, \ldots, x_n$ of C/θ so that

$$f(T^{\bar{C}/\theta}(x_1, \ldots, x_n)) \neq T^{\bar{D}/\sigma}(f(x_1), \ldots, f(x_n)).$$

Since each of these algebras has at most k elements, we can, by identifying variables, assume that $n \leq k$. By our construction, there is a basic operation S of $\underline{\mathbf{A}}$ so that $S^{\underline{A}} = T^{\bar{A}}$. Then $S^{\bar{C}/\theta} = T^{\underline{C}/\theta}$ and $S^{\underline{D}/\sigma} = T^{\bar{D}/\sigma}$. Hence f does not preserve $S^{\underline{C}/\theta}$, so f fails to be a homomorphism from $\underline{\mathbf{C}}/\theta$ to $\underline{\mathbf{D}}/\sigma$. The reverse implication follows from the fact that $\underline{\mathbf{C}}/\theta$ and $\underline{\mathbf{D}}/\sigma$ are reducts of $\bar{\mathbf{C}}/\theta$ and $\bar{\mathbf{D}}/\sigma$ respectively.

As a result of what we know so far, $\underline{\mathbf{A}}$ is also hereditarily (neutral $\times$ Abelian) in a Maltsev variety.

In order to show that $\bar{\mathbf{A}}$ and $\underline{\mathbf{A}}$ have the same term operations, we show that the subuniverses of $\bar{\mathbf{A}}^n$ are the same as the subuniverses of $\underline{\mathbf{A}}^n$ for all positive integers n. Suppose n is a positive integer and $\underline{\mathbf{B}} \in \mathrm{Sub}\underline{\mathbf{A}}^n$. There are subalgebras $\underline{\mathbf{S}}_1, \ldots, \underline{\mathbf{S}}_n$ of $\underline{\mathbf{A}}$ so that $\underline{\mathbf{B}}$ is a subdirect product of $\prod_{i=1}^n \underline{\mathbf{S}}_i$. For each $i = 1, \ldots, n$, we can select congruences θ_i and τ_i of $\underline{\mathbf{S}}_i$ so that $\underline{\mathbf{S}}_i/\theta_i$ is neutral, $\underline{\mathbf{S}}_i/\tau_i$ is Abelian, and $\underline{\mathbf{S}}_i \cong \underline{\mathbf{S}}_i/\theta_i \times \underline{\mathbf{S}}_i/\tau_i$. Let

$$f : \prod_{i=1}^n S_i \to \left(\prod_{i=1}^n S_i/\theta_i\right) \times \left(\prod_{i=1}^n S_i/\tau_i\right)$$

be the canonical bijection. Then f provides an isomorphism of $\prod_{i=1}^n \underline{\mathbf{S}}_i$ and $(\prod_{i=1}^n \underline{\mathbf{S}}_i/\theta_i) \times (\prod_{i=1}^n \underline{\mathbf{S}}_i/\tau_i)$. From Claims 2 and 5 it follows that each $\bar{\mathbf{S}}_i$ is isomorphic to $\bar{\mathbf{S}}_i/\theta_i \times \bar{\mathbf{S}}_i/\tau_i$ and that f is also an isomorphism of $\prod_{i=1}^n \bar{\mathbf{S}}_i$ and $(\prod_{i=1}^n \bar{\mathbf{S}}_i/\theta_i) \times (\prod_{i=1}^n \bar{\mathbf{S}}_i/\tau_i)$. Let P_N be the projection of $(\prod_{i=1}^n \underline{\mathbf{S}}_i/\theta_i) \times (\prod_{i=1}^n \underline{\mathbf{S}}_i/\tau_i)$ onto $\prod_{i=1}^n \underline{\mathbf{S}}_i/\theta_i$, and let P_A be the projection onto $\prod_{i=1}^n \underline{\mathbf{S}}_i/\tau_i$. Let $\underline{\mathbf{B}}_N = P_N(f(\underline{\mathbf{B}}))$ and $\underline{\mathbf{B}}_A = P_A(f(\underline{\mathbf{B}}))$. (Note that P_N and P_A - as projections - are also homomorphisms out of $(\prod_{i=1}^n \bar{\mathbf{S}}_i/\theta_i) \times (\prod_{i=1}^n \bar{\mathbf{S}}_i/\tau_i)$ onto $\prod_{i=1}^n \bar{\mathbf{S}}_i/\theta_i$ and $\prod_{i=1}^n \bar{\mathbf{S}}_i/\tau_i$.)

We claim that $\mathrm{Clo}(\prod_{i=1}^n \underline{\mathbf{S}}_i/\tau_i) = \mathrm{Clo}(\prod_{i=1}^n \bar{\mathbf{S}}_i/\tau_i)$. Since each $\underline{\mathbf{S}}_i/\tau_i$ is Abelian, so is each $\bar{\mathbf{S}}_i/\tau_i$. Since we are in the presence of a Maltsev term, these algebras are actually affine. Hence the algebras $\prod_{i=1}^n \underline{\mathbf{S}}_i/\tau_i$ and $\prod_{i=1}^n \bar{\mathbf{S}}_i/\tau_i$ are affine. As such, the clones of these two algebras are generated from their ternary term operations. In order to show $\mathrm{Clo}(\prod_{i=1}^n \underline{\mathbf{S}}_i/\tau_i) = \mathrm{Clo}(\prod_{i=1}^n \bar{\mathbf{S}}_i/\tau_i)$, therefore, we simply need to show $\mathrm{Clo}_3(\prod_{i=1}^n \underline{\mathbf{S}}_i/\tau_i) = \mathrm{Clo}_3(\prod_{i=1}^n \bar{\mathbf{S}}_i/\tau_i)$. Suppose T is a ternary term in the language of $\bar{\mathbf{A}}$. We observe that the operation $T^{\prod_{i=1}^n \bar{S}_i/\tau_i}$ is a term operation of $\prod_{i=1}^n \underline{\mathbf{S}}_i/\tau_i$. From the manner in which $\underline{\mathbf{A}}$ was defined, there is a term S in the language of $\underline{\mathbf{A}}$ so that $S^{\underline{A}} = T^{\bar{A}}$. $S^{\underline{A}}$ induces an operation on $\prod_{i=1}^n \underline{\mathbf{S}}_i/\tau_i$ in the usual way. Since $S^{\underline{A}} = T^{\bar{A}}$, it should be clear that $T^{\prod_{i=1}^n \bar{S}_i/\tau_i} = S^{\prod_{i=1}^n \underline{S}_i/\tau_i}$. Thus

$$\mathrm{Clo}_3\left(\prod_{i=1}^n \underline{\mathbf{S}}_i/\tau_i\right) \supseteq \mathrm{Clo}_3\left(\prod_{i=1}^n \bar{\mathbf{S}}_i/\tau_i\right).$$

The proof that the reverse inclusion holds is symmetric. Since $\prod_{i=1}^n \underline{\mathbf{S}}_i/\tau_i$ and $\prod_{i=1}^n \bar{\mathbf{S}}_i/\tau_i$ have the same term operations, and since B_A is a subuniverse of $\prod_{i=1}^n \underline{\mathbf{S}}_i/\tau_i$, it follows that B_A is also the universe of a subalgebra $\bar{\mathbf{B}}_A$ of $\prod_{i=1}^n \bar{\mathbf{S}}_i/\tau_i$.

We now address the neutral factors. Note that since $\underline{\mathbf{B}}$ projects onto each $\underline{\mathbf{S}}_i$, it follows that $\underline{\mathbf{B}}_N$ projects onto each $\underline{\mathbf{S}}_i/\theta_i$. For all $i \neq j \leq n$, let $\alpha_{i,j}^{B_N} : \underline{\mathbf{S}}_i/\theta_i \to \underline{\mathbf{B}_N}/(\eta_i^{B_N} \vee \eta_j^{B_N})$

be as in Lemma 4.15. We will write $\alpha_{i,j}$ for $\alpha_{i,j}^{B_N}$. Now, for each $i \neq j \leq n$, we can find $\underline{\mathbf{D}}_{\{i,j\}} \in \mathrm{Sub}\underline{\mathbf{A}}$ and $\sigma_{\{i,j\}} \in \mathrm{Con}\underline{\mathbf{D}}_{\{i,j\}}$ so that there is an isomorphism

$$\beta_{\{i,j\}} : \underline{\mathbf{B}_N}/(\eta_i^{B_A} \vee \eta_j^{B_A}) \to \underline{\mathbf{D}}_{\{i,j\}}/\sigma_{\{i,j\}}.$$

(If $h : \underline{\mathbf{S}}_i \to \underline{\mathbf{S}}_i/\theta_i$ is the canonical map, then it suffices to let $\underline{\mathbf{D}}_{\{i,j\}} = \underline{\mathbf{S}}_i$ and $\sigma_{\{i,j\}} = \ker(\alpha_{i,j}h)$.) Let $\gamma_{i,j} : \underline{\mathbf{S}}_i/\theta_i \to \underline{\mathbf{D}}_{\{i,j\}}/\sigma_{\{i,j\}}$ be given by $\gamma_{i,j} = \beta_{\{i,j\}}\alpha_{i,j}$. With this arrangement and Lemma 4.15.

$$B_N = \left\{ \langle x_1, \ldots, x_n \rangle \in \prod_{i=1}^{n} S_i/\theta_i : \alpha_{i,j}(x_i) = \alpha_{j,i}(x_j) \forall i \neq j \right\}.$$

But since each $\beta_{\{i,j\}}$ is bijective, we can also write:

$$B_N = \left\{ \langle x_1, \ldots, x_n \rangle \in \prod_{i=1}^{n} S_i/\theta_i : \gamma_{i,j}(x_i) = \gamma_{j,i}(x_j) \forall i \neq j \right\}.$$

Since each $\gamma_{i,j}$ is also a homomorphism from $\bar{\mathbf{S}}_i/\theta_i$ onto $\bar{\mathbf{D}}_{\{i,j\}}/\sigma_{\{i,j\}}$ (Claim 5), B_N is the universe of a subalgebra $\bar{\mathbf{B}}_N$ of $\prod_{i=1}^{n} \bar{\mathbf{S}}_i/\theta_i$ (it is not difficult to show that any subset of a product of algebras defined in this manner using homomorphisms is a subuniverse).

Let $\theta = \ker(P_N f) \vee \ker(P_A f)$. Then $\underline{\mathbf{B}}/\theta$ is neutral and Abelian. Therefore $\underline{\mathbf{B}}/\theta$ must be trivial, so

$$1 = \theta = \ker(P_N f) \vee \ker(P_A f) = \ker(P_N f) \circ \ker(P_A f).$$

Since it is also the case that $\ker P_N f \wedge \ker P_A f = 0_B$, $\underline{\mathbf{B}} \cong \underline{\mathbf{B}}_N \times \underline{\mathbf{B}}_A$ with $P_N f$ and $P_A f$ acting as the projection homomorphisms. This means that for any $x \in A^n$, x is in B if and only if $P_N f(x) \in B_N$ and $P_A f(x) \in B_A$. That is, $B = (P_N f)^{-1}(B_N) \cap (P_A f)^{-1}(B_A)$. Since $P_N f : \bar{\mathbf{A}}^n \to \prod_{i=1}^{n} \bar{\mathbf{S}}_i/\theta_i$ and $P_A f : \bar{\mathbf{A}}^n \to \prod_{i=1}^{n} \bar{\mathbf{S}}_i/\tau_i$ are surjective homomorphisms, it follows that B is a subuniverse of $\bar{\mathbf{A}}^n$.

Thus, every subuniverse of a direct power of $\underline{\mathbf{A}}$ is a subuniverse of a direct power of $\bar{\mathbf{A}}$. Since the reverse inclusion is trivial, we have that the subuniverses of direct powers of $\bar{\mathbf{A}}$ are identical to those of $\underline{\mathbf{A}}$. Because the clones of these algebras can be realized as collections of certain subuniverses of direct powers in the familiar way, it follows that $\mathrm{Clo}\bar{\mathbf{A}} = \mathrm{Clo}\underline{\mathbf{A}}$. □

Since there are only finitely many possibilities for $\mathrm{Clo}_{k+2}\mathbf{A}$ when A is finite, this lemma has the following immediate corollary:

COROLLARY 4.17. *There are finitely many hereditarily (neutral × Abelian) algebras with a Maltsev term on any finite set.*

If **A** is finite and hereditarily (neutral $\times$ Abelian) in a congruence permuting variety, then the previous lemma tells us that the centralizer clone of **A** is determined by $\mathrm{Clo}_{|A|+2}\mathbf{A}$. Theorem 2.1 now gives:

THEOREM 4.18. *Suppose* **A** *is a finite hereditarily (neutral* $\times$ *Abelian) algebra with a Maltsev term. PPC#*$(\mathbf{A}) \leq |A| + 2$.

Of course, this class of algebras may seem a little artificial. The theorem gives us a corollary for algebras in the following class of varieties. A variety $\mathcal{V}$ is **directly representable** if and only if it is finitely generated and has up to isomorphism a finite set of directly indecomposable members - all finite. In [12], R. McKenzie gives the following characterization of when a finite set of finite algebras generates a directly representable variety:

THEOREM 4.19. [12] *Let* $\mathcal{K}$ *be an arbitrary set of similar finite algebras.* $\mathcal{V}(\mathcal{K})$ *is directly representable if and only if the following hold:*

1. $\mathcal{V}(\mathcal{K})$ *has permuting congruences.*
2. *Every member of* $S(\mathcal{K})$ *is isomorphic to a direct product of simple algebras and Abelian algebras.*
3. *The variety generated by the set of Abelian factors of members of* $S(\mathcal{K})$ *is directly representable.*

Since the finite product of finite neutral (or Abelian) algebras in a congruence modular variety is neutral (Abelian), and since a simple algebra is either neutral or Abelian, any finite algebra which generates a directly representable variety is hereditarily (neutral $\times$ Abelian). Hence:

COROLLARY 4.20. *Suppose* **A** *is a finite algebra which generates a directly representable variety. Then PPC#*$(\mathbf{A}) \leq |A| + 2$.

5. Closing remarks

In this paper we have seen how the structure of an algebra **A** can influence the size of the generating set of PPC(Clo**A**). Theorem 3.1 insures that our observations about this influence are not trivial. It would be interesting to further investigate the relationship between the structure of a finite algebra **A** and its primitive positive clone. For example, every Maltsev condition satisfied by the variety generated by **A** passes on to the variety generated by $\langle A, \mathrm{PPC}(\mathrm{Clo}\mathbf{A})\rangle$, while the reverse may not be true. What other properties of **A** are passed on to $\langle A, \mathrm{PPC}(\mathrm{Clo}\mathbf{A})\rangle$? Which properties of $\langle A, \mathrm{PPC}(\mathrm{Clo}\mathbf{A})\rangle$ are forced upon **A**? In particular, it would be interesting to investigate the structure of $\langle A, \mathrm{PPC}(\mathrm{Clo}\mathbf{A})\rangle$ for a familiar algebra **A**. For example, if **A** is a finite group or a lattice, what can be said about $\langle A, \mathrm{PPC}(\mathrm{Clo}\mathbf{A})\rangle$? What if **A** is Abelian or even affine?

While investigating the relationship between **A** and PPC(**A**), it should be worthwhile to include **Z**(**A**), as this primitive positive clone also reflects the structure of **A**. For example, it is easy to see that **A** is idempotent if and only if **Z**(**A**) contains all of the constant operations. Also, if **A** is a strictly simple Maltsev algebra, then **Z**(**A**) determines **A** up to categorical equivalence [15].

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