Algebra univers. **54** (2005) 65–71
0002-5240/05/010065 – 07
DOI 10.1007/s00012-005-1922-4

Algebra Universalis

Subdirect products of hereditary congruence lattices

John W. Snow

Abstract. A congruence lattice $\mathbf{L}$ of an algebra $\mathbf{A}$ is called power-hereditary if every 0-1 sublattice of $\mathbf{L}^n$ is the congruence lattice of an algebra on A^n for all positive integers n. Let $\mathbf{A}$ and $\mathbf{B}$ be finite algebras. We prove

- If Con$\mathbf{A}$ is distributive, then every subdirect product of Con$\mathbf{A}$ and Con$\mathbf{B}$ is a congruence lattice on $A \times B$.
- If Con$\mathbf{A}$ is distributive and Con$\mathbf{B}$ is power-hereditary, then $(\mathrm{Con}\mathbf{A}) \times (\mathrm{Con}\mathbf{B})$ is power-hereditary.
- If Con$\mathbf{A} \cong \mathbf{N}_5$ and Con$\mathbf{B}$ is modular, then every subdirect product of Con$\mathbf{A}$ and Con$\mathbf{B}$ is a congruence lattice.
- Every congruence lattice representation of $\mathbf{N}_5$ is power-hereditary.

1. Introduction

A finite lattice is representable if it is isomorphic to the congruence lattice of a finite algebra. If $\mathbf{L}$ is the congruence lattice of a finite algebra $\mathbf{A}$ and every 0-1 sublattice of $\mathbf{L}$ is also the congruence lattice of an algebra with the same universe as $\mathbf{A}$, then $\mathbf{L}$ is called a hereditary congruence lattice. Furthermore, if every 0-1 sublattice of $\mathbf{L}^n$ is the congruence lattice of an algebra on A^n for every positive integer n, then $\mathbf{L}$ is a power-hereditary congruence lattice.

In [6] the author proves that the lattice of equivalence relations on a three element set (which is isomorphic to $\mathbf{M}_3$) is a power-hereditary congruence lattice thereby proving that every finite lattice in the variety generated by $\mathbf{M}_3$ is representable. Hegedűs and Pálfy in [2] improve upon this result by proving that every finite lattice in the variety generated by all finite lattices formed by gluing together copies of $\mathbf{M}_3$ in certain ways is representable. They then introduce the notion of a (power-) hereditary congruence lattice and give examples of congruence lattices which are power-hereditary, hereditary but not power-hereditary, and not hereditary. In this paper we investigate subdirect products of congruence lattices in which one of the lattices is distributive or is isomorphic to $\mathbf{N}_5$.

Presented by E. W. Kiss.
Received November 11, 2004; accepted in final form November 23, 2004.
2000 *Mathematics Subject Classification*: Primary: 08A30; Secondary: 06B15.
Key words and phrases: congruence lattice, primitive positive formula, hereditary congruence lattice.

2. Preliminaries

If α is a binary relation on a set A and β is a binary relation on a set B, then the relation $\langle \alpha, \beta \rangle$ is a binary relation on $A \times B$ defined so that $\langle a_1, b_1 \rangle \langle \alpha, \beta \rangle \langle a_2, b_2 \rangle$ if and only if $a_1 \alpha a_2$ and $b_1 \beta b_2$. If $\mathbf{L}$ is a lattice of equivalence relations on a set A and $\mathbf{M}$ is a lattice of equivalence relations on a set B, then $\mathbf{L} \times \mathbf{M}$ is the set of all equivalence relations on $A \times B$ of the form $\langle \alpha, \beta \rangle$ where $\alpha \in \mathbf{L}$ and $\beta \in \mathbf{M}$. These definitions extend naturally to direct powers $\mathbf{L}^n$ of lattices of equivalence relations. If $\mathbf{L}$ is the congruence lattice of a finite algebra $\mathbf{A}$ and every 0-1 sublattice of $\mathbf{L}$ is also the congruence lattice of an algebra with the same universe as $\mathbf{A}$, then $\mathbf{L}$ is called a *hereditary congruence lattice*. Furthermore, if every 0-1 sublattice of $\mathbf{L}^n$ is the congruence lattice of an algebra on A^n for all positive integers n, then $\mathbf{L}$ is a *power-hereditary congruence lattice*.

By a *representation* or a *congruence representation* of a finite lattice $\mathbf{L}$ we will mean the congruence lattice Con$\mathbf{A}$ of a finite algebra such that Con$\mathbf{A} \cong \mathbf{L}$. If Con$\mathbf{A}$ is a representation of $\mathbf{L}$ and Con$\mathbf{A}$ is a (power-)hereditary congruence lattice, then we will say that Con$\mathbf{A}$ is a (power-)hereditary representation.

A *primitive positive formula* is a formula of the form $\exists \wedge$ (atomic). If Φ is a primitive positive formula employing binary relation symbols $r_1, \ldots, r_n$ and if Φ has two free variables, then Φ naturally induces an operation on the set of binary relations of any set. If $\theta_1, \ldots, \theta_n$ are binary relations on a set A, then we will use $\Phi(\theta_1, \ldots, \theta_n)$ to represent the binary relation on A defined by interpreting each r_i in Φ as θ_i. The operation $\langle \theta_1, \ldots, \theta_n \rangle \mapsto \Phi(\theta_1, \ldots, \theta_n)$ is order preserving, and when it is applied to products of relations can be applied coordinate-wise.

In [5] the author proves that the set of all representable finite lattices is closed under certain lattice theoretic operations. We will employ some of those results here. The main tool exploited there is the following lemma which follows from the fact that a set of relations on a finite set is the set of all relations compatible with an algebra on the set if and only if the relations are closed under primitive positive definitions [1, 3].

Lemma 2.1. ([5] Corollary 2.2) *Suppose* $\mathbf{L}$ *is a* 0-1 *lattice of equivalence relations on a finite set A. There is an algebra* $\mathbf{A}$ *on* A *with* Con$\mathbf{A} = \mathbf{L}$ *if and only if every equivalence relation on* A *which can be defined from* $\mathbf{L}$ *by a primitive positive formula is already in* $\mathbf{L}$.

If A and X are sets, then we will use A^X to represent the set of all functions from X to A. If $\mathbf{P}$ and $\mathbf{Q}$ are posets, then $\mathbf{P}^{\mathbf{Q}}$ is the set of all order preserving maps from $\mathbf{Q}$ to $\mathbf{P}$. Since the operations induced on binary relations by primitive positive formulas are order preserving and apply coordinate-wise, an immediate consequence of Lemma 2.1 is

Corollary 2.2. *Suppose that* $\mathbf{A}$ *is a finite algebra and* $\mathbf{P}$ *is a finite poset. Then* $(\mathrm{Con}\mathbf{A})^{\mathbf{P}}$ *is a congruence lattice on the set* A^P.

Taking $\mathbf{P}$ to be $\{0,1\}$ with the usual order, we get

Corollary 2.3. *Suppose that* $\mathbf{N}$ *is the congruence lattice of a finite algebra and let* $\mathbf{L} = \{\langle u,v\rangle \in \mathbf{N}^2 : u \leq v\}$. *Then* $\mathbf{L}$ *is a congruence lattice.*

Among the constructions from [5] we will need the following.

Lemma 2.4. ([5] Lemma 3.2) *Suppose* $\mathbf{A}$ *is a finite algebra and* α *and* β *are equivalence relations on* A. *There is an algebra* $\mathbf{A}'$ *on* $\mathbf{A}$ *with*

$$\mathrm{Con}\mathbf{A}' = \{x \in \mathrm{Con}\mathbf{A} : x \leq \alpha \text{ or } x \geq \beta\}.$$

In [2] Hegedűs and Pálfy give the following characterization of sublattices of products of lattices as intersections of sublattices of a special form.

Lemma 2.5. ([2] Lemma 4.7) *Let* $\mathbf{L}_1$ *and* $\mathbf{L}_2$ *be arbitrary lattices and* $\mathbf{L} \subseteq \mathbf{L}_1 \times \mathbf{L}_2$ *a sublattice. Let us define*

$$\begin{aligned}
\mathbf{L}'_1 &= \{x \in \mathbf{L}_1 : (\exists b \in \mathbf{L}_2)\langle x,b\rangle \in \mathbf{L}\}\\
\mathbf{L}'_2 &= \{y \in \mathbf{L}_2 : (\exists a \in \mathbf{L}_1)\langle a,y\rangle \in \mathbf{L}\}\\
\mathbf{L}^*_1 &= \{\langle x,y\rangle \in \mathbf{L}_1 \times \mathbf{L}_2 : (\exists \langle a,b\rangle \in \mathbf{L})(x \leq a \text{ and } b \leq y)\}\\
\mathbf{L}^*_2 &= \{\langle x,y\rangle \in \mathbf{L}_1 \times \mathbf{L}_2 : (\exists \langle a,b\rangle \in \mathbf{L})(x \geq a \text{ and } b \geq y)\}
\end{aligned}$$

Then $\mathbf{L} = (\mathbf{L}'_1 \times \mathbf{L}_2) \cap (\mathbf{L}_1 \times \mathbf{L}'_2) \cap \mathbf{L}^*_1 \cap \mathbf{L}^*_2$. *Moreover, if* $\mathbf{L}$ *is a* 0-1 *sublattice, then* $\mathbf{L}^*_1$ *and* $\mathbf{L}^*_2$ *are subdirect products in* $\mathbf{L}_1 \times \mathbf{L}_2$ *with* $\langle 0,1\rangle \in \mathbf{L}^*_1$ *and* $\langle 1,0\rangle \in \mathbf{L}^*_2$.

Using this lemma and the fact that lattices have a majority term, Hegedűs and Pálfy prove (Eq(X) is the lattice of all equivalence relations on the set X):

Lemma 2.6. ([2] Theorem 4.8) *Let* X *be a finite set and* $\mathbf{L} \subseteq \mathrm{Eq(X)}$ *a* 0-1 *sublattice. If every subdirect product* $\mathbf{L}'' \subseteq \mathbf{L}\times\mathbf{L} \subseteq \mathrm{Eq(X^2)}$ *containing* $(\{0\}\times\mathbf{L})\cup(\mathbf{L}\times\{1\})$ *is a congruence lattice, then* $\mathbf{L} \subseteq \mathrm{Eq(X)}$ *is a power hereditary congruence lattice.*

3. Products with Distributive Lattices

Every 0-1 distributive lattice of equivalence relations on a finite set is a congruence lattice [4], so every distributive congruence lattice is power-hereditary. In this section we will consider sublattices of direct products of distributive congruence lattices with other congruence lattices. Let $\mathbf{N}$ and $\mathbf{M}$ be finite lattices and $\mathbf{L}$ a subdirect product of $\mathbf{N}$ and $\mathbf{M}$. For any $x \in \mathbf{N}$, let

$$x^{\uparrow} = \vee\{y \in \mathbf{M} : \langle x,y\rangle \in \mathbf{L}\} \text{ and}$$
$$x^{\downarrow} = \wedge\{y \in \mathbf{M} : \langle x,y\rangle \in \mathbf{L}\}.$$

The following lemma is not difficult and will be used frequently without reference.

Lemma 3.1. *Let* $\mathbf{N}$ *and* $\mathbf{M}$ *be finite lattices and* $\mathbf{L}$ *a subdirect product of* $\mathbf{N}$ *and* $\mathbf{M}$.

(1) *For any* $x \in \mathbf{N}$ *and* $y \in \mathbf{M}$, $\langle x, y\rangle \in \mathbf{L}$ *if and only if* $x^{\downarrow} \leq y \leq x^{\uparrow}$.
(2) *The map* $x \mapsto x^{\downarrow}$ *is a join homomorphism.*
(3) *The map* $x \mapsto x^{\uparrow}$ *is a meet homomorphism.*

Suppose that $\mathbf{L}$ is a finite distributive lattice. Denote the set of meet prime elements of $\mathbf{L}$ as $\mathrm{M}(\mathbf{L})$. For any $x \in \mathrm{M}(\mathbf{L})$, let $\bar{x} = \wedge\{y \in \mathbf{L} : y \not\leq x\}$. Then $\bar{x}$ is join prime and $\mathbf{L}$ is the disjoint union of the intervals $[0, x]$ and $[\bar{x}, 1]$.

Theorem 3.2. *Suppose that* $\mathbf{D} = \mathrm{Con}\mathbf{A}$ *and* $\mathbf{M} = \mathrm{Con}\mathbf{B}$ *are congruence lattices of finite algebras with* $\mathbf{D}$ *distributive. Then every subdirect product of* $\mathbf{D}$ *and* $\mathbf{M}$ *is the congruence lattice of an algebra on* $A \times B$.

Proof. Let $\mathbf{L}$ be a subdirect product of $\mathbf{D}$ and $\mathbf{M}$. Let

$$\mathbf{N} = \bigcap_{x \in \mathrm{M}(\mathbf{D})} \{\langle u, v\rangle \in \mathbf{D} \times \mathbf{M} : \langle u, v\rangle \leq \langle x, x^{\uparrow}\rangle \text{ or } \langle u, v\rangle \geq \langle \bar{x}, \bar{x}^{\downarrow}\rangle\}$$

We claim that $\mathbf{L} = \mathbf{N}$. It will then follow that $\mathbf{L}$ is a congruence lattice by Lemma 2.4. Let $\langle u, v\rangle \in \mathbf{L}$ and $x \in \mathrm{M}(\mathbf{D})$. Either $u \leq x$ — in which case $v \leq x^{\uparrow}$ by Lemma 3.1 — or $u \geq \bar{x}$ — in which case $\bar{x}^{\downarrow} \leq v$. Thus, either $\langle u, v\rangle \leq \langle x, x^{\uparrow}\rangle$ or $\langle u, v\rangle \geq \langle \bar{x}, \bar{x}^{\downarrow}\rangle$. This is true for all $x \in \mathrm{M}(\mathbf{D})$, so $\langle u, v\rangle \in \mathbf{N}$.

Next, let $\langle u, v\rangle \in \mathbf{N}$. Since $u \in \mathbf{D}$ and since $\mathbf{D}$ is a finite distributive lattice, there exist $m_1, \ldots, m_s \in \mathrm{M}(\mathbf{D})$ with $u = m_1 \wedge m_2 \wedge \cdots \wedge m_s$. Also, there are $n_1, n_2, \ldots, n_t \in \mathrm{M}(\mathbf{D})$ so that $u = \bar{n}_1 \vee \bar{n}_2 \vee \cdots \vee \bar{n}_t$. Since $\langle u, v\rangle \in \mathbf{N}$, for each $i = 1, 2, \ldots, s$ it must be that $\langle u, v\rangle \leq \langle m_i, {m_i}^{\uparrow}\rangle$ (since $u \leq m_i$). Also, for $i = 1, 2, \ldots, t$ we have $\langle u, v\rangle \geq \langle \bar{n}_i, {\bar{n}_i}^{\downarrow}\rangle$ (since $u \geq \bar{n}_i$). Therefore

$$u^{\downarrow} = {\bar{n}_1}^{\downarrow} \vee {\bar{n}_2}^{\downarrow} \vee \cdots \vee {\bar{n}_t}^{\downarrow} \leq v \leq {m_1}^{\uparrow} \wedge {m_2}^{\uparrow} \wedge \cdots \wedge {m_s}^{\uparrow} = u^{\uparrow}.$$

Hence $\langle u, v\rangle \in \mathbf{L}$, and we have $\mathbf{L} = \mathbf{N}$. It follows now from Lemma 2.4 that $\mathbf{L}$ is indeed a congruence lattice. □

Note that this proof only required that $\mathbf{M}$ be a congruence lattice, not a hereditary congruence lattice.

Theorem 3.3. *Suppose that* $\mathbf{A}$ *and* $\mathbf{B}$ *are finite algebras with* $\mathrm{Con}\mathbf{A}$ *distributive and* $\mathrm{Con}\mathbf{B}$ *power-hereditary. Then* $\mathrm{Con}\mathbf{A} \times \mathrm{Con}\mathbf{B}$ *is a power-hereditary congruence lattice.*

Proof. Let $\mathbf{D} = \mathrm{Con}\mathbf{A}$ and $\mathbf{M} = \mathrm{Con}\mathbf{B}$. By Lemma 2.6, it suffices to show that every subdirect product $\mathbf{L} \subseteq (\mathbf{D} \times \mathbf{M}) \times (\mathbf{D} \times \mathbf{M})$ is a congruence lattice. Let $\mathbf{L}$ be such a lattice. $\mathbf{L}$ is a lattice of equivalence relations on the set $(A \times B) \times (A \times B)$. There is a natural bijection between $(A \times B) \times (A \times B)$ and $A^2 \times B^2$. Under this bijection, $\mathbf{L}$ corresponds to a sublattice $\mathbf{L}'$ of $\mathbf{D}^2 \times \mathbf{M}^2$. $\mathbf{L}$ is a congruence lattice if and only if $\mathbf{L}'$ is. The projection $\mathbf{L}'_D$ of $\mathbf{L}'$ to $\mathbf{D}^2$ is a distributive lattice of equivalence relations and is therefore a congruence lattice on A^2. On the other hand, the projection $\mathbf{L}'_M$ of $\mathbf{L}'$ to $\mathbf{M}^2$ is a congruence lattice on B^2 because $\mathbf{M}$ is power-hereditary. $\mathbf{L}'$ is then a subdirect product of a distributive congruence lattice $\mathbf{L}'_D$ and another congruence lattice $\mathbf{L}'_M$. By Theorem 3.2, $\mathbf{L}'$ is a congruence lattice, and then so is $\mathbf{L}$. It now follows from Lemma 2.6 that $\mathbf{D} \times \mathbf{M}$ is a power-hereditary congruence lattice. □

4. Representations of $\mathbf{N}_5$

In this section, we will consider subdirect products of representations of $\mathbf{N}_5$ and representations of other finite lattices. Assume that $\mathbf{N} = \mathrm{Con}\mathbf{A}$ and $\mathbf{M} = \mathrm{Con}\mathbf{B}$ are congruence lattices of finite algebras with $\mathbf{N} \cong \mathbf{N}_5$. Let $\mathbf{L} \subseteq \mathbf{N} \times \mathbf{M}$ be a subdirect product. Assume that $\mathbf{N} = \{0, 1, a, b, c\}$ with 0 and 1 the minimal and maximal elements and $c \prec b$ the critical cover. We will also refer to the minimal and maximal elements of $\mathbf{M}$ as 0 and 1.

Lemma 4.1. *If $\mathbf{L}$ contains $\{0\} \times \mathbf{M}$ and $\mathbf{N} \times \{1\}$ and if $a^{\downarrow}$ and $b^{\downarrow}$ are comparable or $c^{\downarrow} = b^{\downarrow}$ then $\mathbf{L}$ is the congruence lattice of an algebra on $A \times B$.*

Proof. Suppose first that $c^{\downarrow} = b^{\downarrow}$. Then $\mathbf{L} = X \cap Y$ where

$$X = \{\langle u, v\rangle : \langle u, v\rangle \geq \langle a, a^{\downarrow}\rangle \text{ or } \langle u, v\rangle \leq \langle b, 1\rangle\} \text{ and}$$
$$Y = \{\langle u, v\rangle : \langle u, v\rangle \geq \langle c, c^{\downarrow}\rangle \text{ or } \langle u, v\rangle \leq \langle a, 1\rangle\}.$$

By Lemma 2.4 X and Y are congruence lattices of algebras on $A \times B$. Since $\mathbf{L} = X \cap Y$, $\mathbf{L}$ is also a congruence lattice.

From now on suppose that $c^{\downarrow} \neq b^{\downarrow}$. Assume $a^{\downarrow} \leq b^{\downarrow}$, then $\mathbf{L} = X \cap Y$ where

$$X = \{\langle u, v\rangle : \langle u, v\rangle \geq \langle c, c^{\downarrow}\rangle \text{ or } \langle u, v\rangle \leq \langle a, 1\rangle\} \text{ and}$$
$$Y = \{\langle u, v\rangle : \langle u, v\rangle \geq \langle 0, a^{\downarrow}\rangle \text{ or } \langle u, v\rangle \leq \langle c, 1\rangle\}.$$

(Note here that since $a \vee c = a \vee b$, then $a^{\downarrow} \vee c^{\downarrow} = a^{\downarrow} \vee b^{\downarrow} = b^{\downarrow}$.) Finally, if $a^{\downarrow} > b^{\downarrow}$, then $\mathbf{L} = X \cap Y \cap Z$ where

$$X = \{\langle u, v\rangle : \langle u, v\rangle \geq \langle a, a^{\downarrow}\rangle \text{ or } \langle u, v\rangle \leq \langle b, 1\rangle\},$$
$$Y = \{\langle u, v\rangle : \langle u, v\rangle \geq \langle 0, c^{\downarrow}\rangle \text{ or } \langle u, v\rangle \leq \langle 0, 1\rangle\}, \text{ and}$$
$$Z = \{\langle u, v\rangle : \langle u, v\rangle \geq \langle 0, b^{\downarrow}\rangle \text{ or } \langle u, v\rangle \leq \langle c, 1\rangle\}.$$

In all cases, $\mathbf{L}$ is a congruence lattice. □

A similar argument shows

Lemma 4.2. *If $\mathbf{L}$ contains $\{1\} \times \mathbf{M}$ and $\mathbf{N} \times \{0\}$ and if $a^{\uparrow}$ and $c^{\uparrow}$ are comparable or $c^{\uparrow} = b^{\uparrow}$ then $\mathbf{L}$ is the congruence lattice of an algebra on $A \times B$.*

Corollary 4.3. *If* $\mathbf{B} = \mathbf{A}$ *(so* $\mathbf{M} = \mathbf{N} \cong \mathbf{N}_5$*), then every* 0-1 *sublattice of* $\mathbf{N} \times \mathbf{N}$ *containing* $\{0\} \times \mathbf{N}$ *and* $\mathbf{N} \times \{1\}$ *is the congruence lattice of an algebra on* A^2*.*

Proof. By Lemma 4.1, we need only address the case when $a^{\downarrow}$ is not comparable to $b^{\downarrow}$ and $c^{\downarrow} \neq b^{\downarrow}$. We have that $a^{\downarrow} \vee b^{\downarrow} = a^{\downarrow} \vee c^{\downarrow} = 1^{\downarrow}$, that $1^{\downarrow} \neq a^{\downarrow}, b^{\downarrow}$, and that $c^{\downarrow} < b^{\downarrow}$. Therefore, it must be that case that $c^{\downarrow}$ is also not comparable to $a^{\downarrow}$. The only way to have this arrangement (since $\mathbf{N} \cong \mathbf{N}_5$) is if $x^{\downarrow} = x$ for all $x \in \mathbf{N}$. This means that $\mathbf{L} = \{\langle u, v\rangle \in \mathbf{N}^2 : u \leq v\}$. Applying Corollary 2.3 now tells us that $\mathbf{L}$ is a congruence lattice. □

This and Lemma 2.6 now give us

Theorem 4.4. *Every representation of* $\mathbf{N}_5$ *as the congruence lattice of a finite algebra is a power-hereditary representation.*

Lemmas 4.1 and 4.2 also give

Corollary 4.5. *If* $\mathbf{M}$ *is modular and if* $\mathbf{L}$ *contains either* $(\{0\} \times \mathbf{M}) \cup (\mathbf{N} \times \{1\})$ *or* $(\mathbf{N} \times \{0\}) \cup (\{1\} \times \mathbf{M}))$ *then* $\mathbf{L}$ *is a congruence lattice.*

Proof. Suppose that $\mathbf{L}$ contains $(\{0\} \times \mathbf{M}) \cup (\mathbf{N} \times \{1\})$. As in the proof of Corollary 4.3, we can assume that $a^{\downarrow}$ is not comparable to either $b^{\downarrow}$ or $c^{\downarrow}$ and that $c^{\downarrow} < b^{\downarrow}$.

Since $\mathbf{M}$ is modular, it cannot be that $a^{\downarrow} \wedge b^{\downarrow} = a^{\downarrow} \wedge c^{\downarrow}$ (this would give us a copy of $\mathbf{N}_5$ in $\mathbf{M}$). Also by modularity, it must be that $(a^{\downarrow} \wedge b^{\downarrow}) \vee c^{\downarrow} = b^{\downarrow}$. It now follows that $\mathbf{L} = X \cap Y \cap Z$ where

$$\begin{aligned} X &= \{\langle u, v\rangle : \langle u, v\rangle \geq \langle a, a^{\downarrow}\rangle \text{ or } \langle u, v\rangle \leq \langle b, 1\rangle\}, \\ Y &= \{\langle u, v\rangle : \langle u, v\rangle \geq \langle c, c^{\downarrow}\rangle \text{ or } \langle u, v\rangle \leq \langle a, 1\rangle\}, \text{ and} \\ Z &= \{\langle u, v\rangle : \langle u, v\rangle \geq \langle 0, a^{\downarrow} \wedge b^{\downarrow}\rangle \text{ or } \langle u, v\rangle \leq \langle c, 1\rangle\}. \end{aligned}$$

The proof of the other half of the lemma is similar. □

Together with Lemma 2.5, this proves

Theorem 4.6. *If* $\mathbf{A}$ *and* $\mathbf{B}$ *are finite algebras with* $\mathrm{Con}\mathbf{A} \cong \mathbf{N}_5$ *and* $\mathrm{Con}\mathbf{B}$ *modular, then every subdirect product of* $\mathrm{Con}\mathbf{A}$ *and* $\mathrm{Con}\mathbf{B}$ *is the congruence lattice of an algebra on* $A \times B$*.*

Note that we did not need for $\mathrm{Con}\mathbf{B}$ to be hereditary here.

References

[1] V. G. Bodnarchuk, L. A. Kalužnin, V. N. Kotov and B. A. Romov, *Galois theory for Post algebras I; II*, Kibernetika (Kiev) **3** 1969, 1–10; **5** (1969), 1–9.

[2] P. Hegedűs and P. P. Pálfy, *Finite modular congruence lattices*, preprint, 2004.

[3] R. Pöschel and L. A. Kalužnin, *Funktionen-und Relationenalgebren*, Birkhäuser, Basel, 1979.

[4] R. W. Quackenbush and B. Wolk, *Strong representations of congruence lattices*, Algebra Universalis **1** (1971), 165–166.

[5] J. W. Snow, *A constructive approach to the finite congruence lattice representation problem*, Algebra Universalis **43** (2000), 279–293.
[6] J. W. Snow, *Every finite lattice in $\mathcal{V}(\mathbf{M}_3)$ is representable*, Algebra Universalis **50** (2003), 75–81.

John W. Snow
Department of Mathematics and Statistics, Sam Houston State University, Huntsville, Texas 77341-2206, USA
e-mail: jsnow@shsu.edu