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OPC Lattices and Congruence Heredity

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Abstract. We prove that if **A** is a finite algebra which satisfies a nontrivial idempotent Mal'cev condition, and if Con**A** contains a copy of an order polynomially complete lattice other than **2**, $\mathbf{M}_3$, or $\mathrm{Con}(\mathbb{Z}_2^3)$, then Con**A** is not hereditary.

1. Introduction

A lattice **L** is *order polynomially complete* (OPC) if every order preserving operation on **L** is a polynomial of **L**. The proof of Theorem 5.2 in [7] establishes the following:

Theorem 1.1. *Suppose that* **A** *is a finite algebra so that* $\mathbf{L} = \mathrm{Con}\mathbf{A}$ *is OPC. Then every diagonal sublattice of* $\mathbf{L}^n$ *is the congruence lattice of an algebra on the universe of* $\mathbf{A}^n$.

In [8], the ideas used to prove Theorem 1.1 were employed to prove the following:

Theorem 1.2. *Let* **L** *be the lattice of equivalence relations on the set* $A = \{0, 1, 2\}$. *Every* 0-1 *sublattice of* $\mathbf{L}^n$ *is a congruence lattice on* A^n.

From this theorem, it followed that every finite lattice in the variety generated by $\mathbf{M}_3$ is isomorphic to the congruence lattice of a finite algebra. This result led Hegedűs and Pálfy in [2] to define the notions of congruence heredity and congruence power-heredity. The congruence lattice **L** of a finite algebra **A** is hereditary if every 0-1 sublattice of **L** is the congruence lattice of an algebra with the same universe as **A**. **L** is power-hereditary if every 0-1 sublattice of $\mathbf{L}^n$ is the congruence lattice of an algebra with the same universe as $\mathbf{A}^n$. **A** is congruence (power-)hereditary if Con**A** is (power-)hereditary. In [2], Hegedűs and Pálfy characterized which finite Abelian p-groups have (power-)hereditary congruence lattices. In [6], this author proved that every representation of $\mathbf{N}_5$ as the congruence lattice of a finite algebra is power-hereditary. In [4], Pálfy proved that the lattice $\mathbf{M}_3$ does not share this property

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by giving and example of a finite algebra whose congruence lattice is isomorphic to $\mathbf{M}_3$ but is not power-hereditary. To date, there is no known congruence lattice representation of $\mathbf{M}_4$ which is even hereditary.

Theorem 1.1 could be interpreted as saying that every OPC congruence lattice is somewhat *weakly* power-hereditary. In contrast to this, we prove in this paper that if $\mathbf{A}$ is a finite algebra in a variety which satisfies a nontrivial idempotent Mal'cev condition and if Con$\mathbf{A}$ contains a copy of an OPC lattice other than the two element lattice, $\mathbf{M}_3$, or $\mathrm{Con}(\mathbb{Z}_2^3)$, then Con$\mathbf{A}$ cannot be hereditary. In the process, we give a characterization of congruence (power-)hereditary vector spaces.

2. Preliminaries

By a *representation* or a *congruence representation* of a finite lattice $\mathbf{L}$ we will mean the congruence lattice Con$\mathbf{A}$ of a finite algebra $\mathbf{A}$ such that $\mathrm{Con}\mathbf{A} \cong \mathbf{L}$. A *primitive positive formula* is a formula of the form $\exists \wedge$ (atomic). If Φ is a primitive positive formula employing binary relation symbols $r_1, \ldots, r_n$ and if Φ has two free variables, then Φ naturally induces an operation on the set of binary relations of any set. If $\theta_1, \ldots, \theta_n$ are binary relations on a set A, then we will use $\Phi(\theta_1, \ldots, \theta_n)$ to represent the binary relation on A defined by interpreting each r_i in Φ as θ_i. The operation $\langle \theta_1, \ldots, \theta_n \rangle \mapsto \Phi(\theta_1, \ldots, \theta_n)$ is order preserving, and when it is applied to products of relations can be applied coordinate-wise. Closure under primitive positive definitions characterizes those lattices of equivalence relations that are congruence lattices.

Lemma 2.1. (Corollary 2.2 of [7]) *Suppose* $\mathbf{L}$ *is a* 0-1 *lattice of equivalence relations on a finite set* A*. There is an algebra* $\mathbf{A}$ *on* A *with* $\mathrm{Con}\mathbf{A} = \mathbf{L}$ *if and only if every equivalence relation on* A *which can be defined from* $\mathbf{L}$ *by a primitive positive formula is already in* $\mathbf{L}$*.*

We will assume from here on that **every primitive positive formula only contains binary relation symbols and has exactly two free variables**. We will be interpreting the relations in these formulas only as equivalence relations. Suppose that Φ is any such primitive positive formula and that $r_1, \ldots, r_n$ are the relations symbols in Φ (or are relations interpreted as the symbols in Φ). Let $x_1, \ldots, x_m$ be the variables in Φ. By the *graph of* $\Phi(r_1, \ldots, r_n)$ we will mean the undirected graph $\mathbf{G}$ with vertices $\{x_1, \ldots, x_m\}$ so that for each occurrence of $r_i(x_j, x_k)$ in Φ, there is an edge in $\mathbf{G}$ labelled by r_i between x_j and x_k. A primitive positive formula Φ will be called *connected* if the corresponding graph is connected. If a primitive positive formula Φ is not connected, then its value is completely determined by the component containing the free variables. Thus in Lemma 2.1, **it is sufficient to consider connected primitive positive formulas**. In light of this, we will assume that **all primitive positive formulas are connected**. This assumption of connectedness is something of a formality that can almost be ignored in this paper. If the two free variables in a primitive positive formula are not contained in the same component, then the formula can only define the universal relation. The operations referred to in Lemmas 2.3 and 2.4 would then be the constant 1. By assuming our formulas are connected, we avoid needing this constant operation. The operations in Lemmas 2.3 and 2.4 can then be terms rather than polynomials.

Among the constructions from [7] we will also need the following.

Lemma 2.2. (Lemma 3.2 of [7]) *Suppose* $\mathbf{A}$ *is a finite algebra and* α *and* β *are equivalence relations on* A*. There is an algebra* $\mathbf{A}'$ *on the universe of* $\mathbf{A}$ *with*

$$\mathrm{Con}\mathbf{A}' = \{x \in \mathrm{Con}\mathbf{A} : x \leq \alpha \text{ or } x \geq \beta\}.$$

Congruence heredity and power-heredity are related to how well primitive positive definitions can be interpolated by lattice terms:

Lemma 2.3. (Lemma 2.3 of [5]) *The congruence lattice of a finite algebra* $\mathbf{A}$ *is hereditary if and only if for every primitive positive formula* $\Phi(x_1, \ldots, x_n)$ *and for all* $r_1, \ldots, r_n \in \mathrm{Con}\mathbf{A}$ *if* $\Phi(r_1, \ldots, r_n)$ *is an equivalence relation, then there is a lattice term* $T(x_1, \ldots, x_n)$ *so that* $T(r_1, \ldots, r_n) = \Phi(r_1, \ldots, r_n)$*.*

Lemma 2.4. (Lemma 4.5 of [2]) *The congruence lattice of a finite algebra* $\mathbf{A}$ *is power-hereditary if and only if for every primitive positive formula* $\Phi(x_1, \ldots, x_n)$ *there is a lattice term* $T(x_1, \ldots, x_n)$ *so that if* $r_1, \ldots, r_n \in \mathrm{Con}\mathbf{A}$ *and* $\Phi(r_1, \ldots, r_n)$ *is an equivalence relation, then* $T(r_1, \ldots, r_n) = \Phi(r_1, \ldots, r_n)$*.*

Suppose that $\mathbf{A}$ and $\mathbf{B}$ are finite algebras and $f\colon \mathrm{Con}\mathbf{A} \to \mathrm{Con}\mathbf{B}$ is any function. We will say that f preserves primitive positive definitions if whenever $\Phi(x_1, \ldots, x_n)$ is a primitive positive formula and $r_1, \ldots, r_n \in \mathrm{Con}\mathbf{A}$ so that $\Phi(r_1, \ldots, r_n)$ and $\Phi(f(r_1), \ldots, f(r_n))$ are equivalence relations, then

$$f(\Phi(r_1, \ldots, r_n)) = \Phi(f(r_1), \ldots, f(r_n)).$$

Note that any such function must be a lattice homomorphism. We first observe that such a function must also preserve congruence (power-)heredity.

Lemma 2.5. *Suppose that* $\mathbf{A}$ *and* $\mathbf{B}$ *are finite algebras and that* $f\colon \mathrm{Con}\mathbf{A} \to \mathrm{Con}\mathbf{B}$ *is a surjection which preserves primitive positive definitions. Then if* $\mathrm{Con}\mathbf{A}$ *is (power-)hereditary then* $\mathrm{Con}\mathbf{B}$ *is also (power-)hereditary.*

Proof. Suppose that $\mathrm{Con}\mathbf{A}$ is hereditary. We will show that $\mathrm{Con}\mathbf{B}$ is hereditary. Let Φ be a primitive positive formula and $s_1, \ldots, s_n \in \mathrm{Con}\mathbf{B}$ with $\Phi(s_1, \ldots, s_n)$

an equivalence relation. Find $r_1, \ldots, r_n \in \mathrm{Con}\mathbf{A}$ so that $f(r_i) = s_i$ for all i. Let $\Phi'(x_1, x_2)$ be defined by $\Phi(x_1, x_2) \wedge \Phi(x_2, x_1)$, and define $\Phi''(x_1, x_2)$ by

$$\exists y_0, \ldots, y_{m+1} \Big[\big(\bigwedge_{i=0}^{m} \Phi'(y_i, y_{i+1}) \big) \wedge (y_0 = x_1) \wedge (y_{m+1} = x_2) \Big]$$

where $m = \max(|A|, |B|)$. Then $\Phi''(r_1, \ldots, r_n)$ is an equivalence relation on A and

$$\Phi''(s_1, \ldots, s_n) = \Phi(s_1, \ldots, s_n).$$

(Φ'' is the equivalence relation closure of Φ on any set with m or fewer elements.) Since $\Phi''(r_1, \ldots, r_n)$ is an equivalence relation and $\mathrm{Con}\mathbf{A}$ is hereditary, then by 2.3 there is a lattice term T so that $T(r_1, \ldots, r_n) = \Phi''(r_1, \ldots, r_n)$. Then

$$\begin{aligned} T(s_1, \ldots, s_n) &= T(f(r_1), \ldots, f(r_n)) \\ &= f(T(r_1, \ldots, r_n)) \\ &= f(\Phi''(r_1, \ldots, r_n)) \\ &= \Phi''(f(r_1), \ldots, f(r_n)) \\ &= \Phi''(s_1, \ldots, s_n) \\ &= \Phi(s_1, \ldots, s_n). \end{aligned}$$

Thus $\mathrm{Con}\mathbf{B}$ is hereditary by 2.3. The case for power-heredity is proven similarly. □

Suppose $\mathcal{V}$ is a variety with a set of basic operation symbols F, and suppose $\mathcal{W}$ is any variety. $\mathcal{W}$ is said to *interpret* $\mathcal{V}$ if for every basic operation t of $\mathcal{V}$ there is a $\mathcal{W}$-term s_t so that for every algebra $\mathbf{A} \in \mathcal{W}$ the algebra $\langle A, \{s_t^A : t \in F\}\rangle$ is a member of $\mathcal{V}$. This relationship is denoted by $\mathcal{V} \leq \mathcal{W}$. A variety $\mathcal{V}$ is *finitely presented* if it has a finite set of basic operation symbols and is defined by a finite set of equations. The variety $\mathcal{V}$ is *idempotent* if every basic operation $t(x_1, \ldots, x_n)$ of $\mathcal{V}$ satisfies the equation $t(x, \ldots, x) \approx x$.

Suppose that $\mathcal{V}$ is a finitely presented variety and $\mathcal{W}$ is any variety. The assertion that $\mathcal{W}$ interprets $\mathcal{V}$ is called a *strong Mal'cev condition.* Suppose that

$$\cdots \leq \mathcal{V}_3 \leq \mathcal{V}_2 \leq \mathcal{V}_1$$

are finitely presented varieties. The assertion that $\mathcal{W}$ interprets one of the $\mathcal{V}_i$ is a *Mal'cev condition.* An *idempotent* Mal'cev condition is one in which all of the defining varieties are idempotent. A *nontrivial* Mal'cev condition is one which is not satisfied by the variety of sets. When we say that an algebra $\mathbf{A}$ satisfies a Mal'cev condition, we mean that the variety generated by $\mathbf{A}$ satisfies the Mal'cev condition.

We will need to refer to $\mathbb{Z}_2$ as a group, as a field, and as a vector space. To avoid confusion, we will use $\mathbb{F}_2$ to represent the field. We will use $\mathbb{Z}_2$ to denote the group and the vector space (over $\mathbb{F}_2$).

3. A minimal amount of tame congruence theory

We will need a little tame congruence theory to establish our results. We outline the essentials we need here. Any unproven results in this section are established in [3]. The reader who is familiar with tame congruence theory should at least refer to 3.2, 3.4, 3.8, and 3.9, which show that the constructions of tame congruence theory preserve congruence (power-)heredity.

Suppose that $\mathbf{A}$ is a finite algebra and U is any subset of $\mathbf{A}$. By $\mathbf{A}|_U$ we will denote the *algebra on U induced by* $\mathbf{A}$. This algebra has universe U. Its operations are all polynomials of $\mathbf{A}$ under which U is closed.

A unary polynomial e of an algebra $\mathbf{A}$ is *idempotent* if the equality $e(e(x)) = e(x)$ holds for all $x \in \mathbf{A}$. If e is an idempotent unary polynomial of $\mathbf{A}$ and $U = e(A)$ then every operation of $\mathbf{A}|_U$ is of the form $e \circ f$ where f is a polynomial of $\mathbf{A}$. For any $\theta \in \mathrm{Con}\mathbf{A}$, $e(\theta) = \theta \cap (U \times U)$ and the map $\theta \to e(\theta)$ is a surjective lattice homomorphism from $\mathrm{Con}\mathbf{A}$ to $\mathrm{Con}\mathbf{A}|_U$. This is Lemma 2.3 of [3]. We need a slightly stronger version of this lemma:

Lemma 3.1. (Lemma 2.3 of [3]) *Suppose that $\mathbf{A}$ is a finite algebra, e is an idempotent unary polynomial of $\mathbf{A}$, and $U = e(\mathbf{A})$. Then e induces a surjective lattice homomorphism from $\mathrm{Con}\mathbf{A}$ to $\mathrm{Con}\mathbf{A}|_U$ which preserves primitive positive definitions.*

Proof. That e induces a lattice surjection is proven in [3]. We need only prove that this surjection preserves primitive positive definitions. Suppose Φ is the primitive positive formula given by

$$\langle x_1, x_2 \rangle \in \Phi(s_1, \ldots, s_n) \leftrightarrow \exists x_3, \ldots, x_m \bigwedge_{i=1}^{p} s_{j_i}(x_{k_i}, x_{l_i}).$$

Let $r_1, \ldots, r_n \in \mathrm{Con}\mathbf{A}$ so that $\Phi(r_1, \ldots, r_n)$ and $\Phi(e(r_1), \ldots, e(r_n))$ are both equivalence relations. That $\Phi(e(r_1), \ldots, e(r_n)) \subseteq e(\Phi(r_1, \ldots, r_n))$ should be clear since $U \subseteq A$, each $e(r_i) \subseteq r_i$, and e is idempotent.

Next, suppose that $\langle u_1, u_2 \rangle \in e(\Phi(r_1, \ldots, r_n))$. This means that there are $x_1, \ldots, x_m \in A$ so that $e(x_i) = u_i$ for $i = 1, 2$ and $r_{j_i}(x_{k_i}, x_{l_i})$ for $i = 1, \ldots, p$. For $i = 3, \ldots, m$, let $u_i = e(x_i)$. Then $\langle u_{k_i}, u_{l_i} \rangle \in e(r_{j_i})$ holds for $i = 1, \ldots, p$. Thus $\langle u_1, u_2 \rangle \in \Phi(e(r_1), \ldots, e(r_n))$. This establishes the reverse inclusion and equality. $\square$

Combining this with Lemma 2.5 immediately gives us the following:

Corollary 3.2. *Suppose that $\mathbf{A}$ is a finite algebra and e is an idempotent unary polynomial of $\mathbf{A}$ with $U = e(A)$. If $\mathbf{A}$ is congruence (power-)hereditary, then so is $\mathbf{A}|_U$.*

If θ is a congruence on an algebra $\mathbf{A}$ and $U \subseteq A$, then we will use $\theta|_U$ to represent $\theta \cap (U \times U)$. The following lemma comes from Lemma 2.4 of [3].

Lemma 3.3. (Lemma 2.4 of [3]) *Suppose that* $\mathbf{A}$ *is a finite algebra and* $\beta \in \mathrm{Con}\mathbf{A}$. *Let* B *be any congruence class of* β *and* $\mathbf{B} = \mathbf{A}|_B$. *Then the map* $\theta \to \theta|_B$ *is a lattice homomorphism from the interval* $[0, \beta]$ *in* $\mathrm{Con}\mathbf{A}$ *onto* $\mathrm{Con}\mathbf{B}$ *which preserves primitive positive definitions.*

Proof. Add operations to $\mathbf{A}$ to form an algebra $\mathbf{A}'$ whose congruence lattice is all congruences less than or equal to β along with the universal relation. (This is possible by Lemma 2.2). Let $b \in B$ be arbitrary. Define $e\colon A \to A$ by

$$e(x) = \begin{cases} x & x \in B \\ b & x \notin B. \end{cases}$$

Then e preserves all of the congruences of $\mathbf{A}'$, so we can assume that it was one of the added operations. Also, e is idempotent and $e(A) = B$. The lemma now follows from Lemma 3.1. □

Combining this with Lemma 2.5 immediately gives us the following:

Corollary 3.4. *Suppose that* $\mathbf{A}$ *is a finite algebra and* $\beta \in \mathrm{Con}\mathbf{A}$. *Let* B *be any congruence class of* β. *If* $\mathbf{A}$ *is congruence (power-)hereditary, then so is* $\mathbf{A}|_B$.

Suppose that $\mathbf{A}$ is a finite algebra and $\alpha < \beta$ are congruences of $\mathbf{A}$. Let $U_{\mathbf{A}}(\alpha, \beta)$ be the set of all sets of the form $f(A)$, where f is a unary polynomial of $\mathbf{A}$ and $f(\beta) \nsubseteq \alpha$. Let $M_{\mathbf{A}}(\alpha, \beta)$ be the set of minimal elements of $U_{\mathbf{A}}(\alpha, \beta)$. These will be called the $\langle \alpha, \beta \rangle$-*minimal sets* of $\mathbf{A}$. If $A \in M_{\mathbf{A}}(\alpha, \beta)$, then $\mathbf{A}$ will be called $\langle \alpha, \beta \rangle$-*minimal*. Any $\langle 0, 1 \rangle$-minimal algebra will be called *minimal*.

Suppose that $\mathbf{L}$ is a finite lattice. $\mathbf{L}$ is 0-1 *simple* if for every nonconstant lattice homomorphism $f\colon \mathbf{L} \to \mathbf{L}'$ and for $x = 0, 1$ the equality $f^{-1}(f(x)) = \{x\}$ holds. If $\mathbf{L}$ is 0-1 simple and every strictly increasing meet endomorphism of $\mathbf{L}$ is constant, then $\mathbf{L}$ is called *tight*. We only need to know that the two element lattice, the lattices $\mathbf{M}_n$, and the congruence lattices of finite vector spaces are tight.

In [9] Wille proved that a finite lattice $\mathbf{L}$ is OPC if and only if $\mathbf{L}$ is simple and the only decreasing join endomorphisms are the identity and the constant 0. Since the polynomial operations of a lattice are identical to the polynomial operations of its dual (if viewed with the same universe), we can dualize this to get that a finite lattice $\mathbf{L}$ is OPC if and only if $\mathbf{L}$ is simple and the only increasing meet homomorphisms of $\mathbf{L}$ are the identity map and the constant 1. Since simplicity implies 0-1 simplicity, we have the following:

Lemma 3.5. *A finite lattice is tight and simple if and only if it is OPC.*

The following lemma extracts the information we will need from Lemmas 2.10, 2.11, and 2.13 of [3].

Lemma 3.6. *Suppose that* $\mathbf{A}$ *is a finite algebra and that* $\alpha < \beta \in \mathrm{Con}\mathbf{A}$ *so that, as a lattice, the interval* $[\alpha, \beta]$ *is tight.*

(1) $\mathbf{A}$ *is* $\langle \alpha, \beta \rangle$*-minimal if and only if for every unary polynomial* f *of* $\mathbf{A}$*, either* f *is a permutation or* $f(\beta) \subseteq f(\alpha)$*.*
(2) *If* $U \in M_{\mathbf{A}}(\alpha, \beta)$*, then there is an idempotent polynomial* e *of* $\mathbf{A}$ *so that* $U = e(A)$*.*
(3) *If* $U \in M_{\mathbf{A}}(\alpha, \beta)$*, then* $\mathbf{A}|_U$ *is* $\langle \alpha|_U, \beta|_U \rangle$*-minimal.*

Suppose that $\mathbf{A}$ is a finite algebra and $\alpha < \beta \in \mathrm{Con}\mathbf{A}$. By an $\langle \alpha, \beta \rangle$-*trace* of $\mathbf{A}$, we mean an equivalence class of $\beta|_U$ which is not contained in an α-equivalence class for some $U \in M_{\mathbf{A}}(\alpha, \beta)$.

Lemma 3.7. (Lemma 2.16 of [3]) *Suppose that* $\mathbf{A}$ *is an* $\langle \alpha, \beta \rangle$*-minimal algebra and* N *is an* $\langle \alpha, \beta \rangle$*-trace of* $\mathbf{A}$*. Then* $\mathbf{A}|_N$ *is* $\langle \alpha|_N, \beta|_N \rangle$*-minimal and* $(\mathbf{A}|_N)/(\alpha|_N)$ *is a minimal algebra.*

We will call the minimal algebra $(\mathbf{A}|_N)/(\alpha|_N)$ here the *minimal algebra associated with* the $\langle \alpha, \beta \rangle$-trace N. We will establish a lemma which allows us to pass congruence (power-)heredity all the way down to the minimal algebras associated with traces. To do so, we need the following lemma.

Lemma 3.8. *Suppose that* $\mathbf{N}$ *is a finite algebra and* $\alpha \in \mathrm{Con}\mathbf{N}$*. If* $\mathbf{N}$ *is congruence (power-)hereditary, then so is* $\mathbf{N}/\alpha$*.*

Proof. Add operations to $\mathbf{N}$ to form an algebra $\mathbf{N}'$ whose congruences are those of $\mathbf{N}$ above α along with the identity relation. (This is possible by Lemma 2.2). Let $a_1, \ldots, a_m$ be representatives of the α-equivalence classes of $\mathbf{N}$. Define $e \colon N \to N$ by $e(x) = a_i$, where $x \alpha a_i$. Then e preserves all of the congruences of $\mathbf{N}'$, so we can assume that it was one of the added operations. Also, e is idempotent, so we can apply Lemma 3.1 to conclude that restriction of $\mathrm{Con}\mathbf{N}'$ to $U = e(N)$ preserves primitive positive defitions. It follows, then, that $\mathrm{Con}\mathbf{N}|_U$ is (power-)hereditary. However, the relational structures $\langle N/\alpha, \mathrm{Con}(\mathbf{N}/\alpha) \rangle$ and $\langle U, \mathrm{Con}(\mathbf{N}|_U) \rangle$ are isomorphic, so $\mathrm{Con}(\mathbf{N}/\alpha)$ is also (power-)hereditary. □

Combining 3.2, 3.4, and 3.8 we now have the following:

Lemma 3.9. *Suppose that* $\mathbf{A}$ *is a finite algebra and* $\alpha < \beta$ *are congruences on* $\mathbf{A}$ *so that* $[\alpha, \beta]$ *is tight. If* $\mathbf{A}$ *is congruence (power-)hereditary, then every minimal algebra associated with any* $\langle \alpha, \beta \rangle$*-trace of* $\mathbf{A}$ *is also congruence (power-)hereditary.*

This lemma indicates that the structure of congruence (power-)hereditary minimal algebras might be important in the study of (power-)hereditary congruence lattices. Corollary 4.11 of [3] characterizes minimal algebras up to their polynomials. Two algebras $\mathbf{A}$ and $\mathbf{B}$ are *polynomially equivalent* if and only if they share the same universe and polynomials.

Theorem 3.10. (See Chapter 4 of [3]) *A finite algebra* $\mathbf{A}$ *is minimal if and only if* $\mathbf{A}$ *is polynomially equivalent to one of the following:*

(1) *a* $\mathbf{G}$*-set for some group* $\mathbf{G}$*,*
(2) *a vector space,*
(3) *the two element Boolean algebra,*
(4) *the two element lattice,*
(5) *the two element semilattice.*

This list of minimal algebras will allow us to specify what the minimal algebras associated to traces of congruence hereditary algebras must look like.

Finally, we need to know how Mal'cev conditions relate to induced algebras on traces.

Lemma 3.11. (Theorem 9.6 of [3]) *For a finite algebra* $\mathbf{A}$*, the following are equivalent:*

(1) *No trace on any finite algebra in the variety generated by* $\mathbf{A}$ *is polynomially equivalent to a* $\mathbf{G}$*-set.*
(2) *The variety generated by* $\mathbf{A}$ *satisfies a nontrivial idempotent Mal'cev condition.*

4. Representing OPC lattices

In this section, we prove that in the presence of an idempotent Mal'cev condition, a copy of an OPC lattice other than a select few in the congruence lattice of an algebra prevents congruence heredity.

Lemma 4.1. *If a finite vector space* $\mathbf{V}$ *has dimension at least two and does not satisfy* $x + x = 0$*, then* $\mathbf{V}$ *is not congruence hereditary (and hence not power-hereditary).*

Proof. This proof is almost identical to the proof of Example 5.1 of [2]. We can assume that for some finite field $\mathbf{F}$, $\mathbf{V}$ is $\mathbf{F}^n$ as an $\mathbf{F}$-vector space with $n \geq 2$. Define the following four congruences on $\mathbf{V}$:

$$\langle a_0, a_1, \ldots, a_{n-1}\rangle \eta_0 \langle b_0, b_1, \ldots, b_{n-1}\rangle \leftrightarrow a_0 = b_0$$

$$\langle a_0, a_1, \ldots, a_{n-1}\rangle \eta_1 \langle b_0, b_1, \ldots, b_{n-1}\rangle \leftrightarrow a_1 = b_1$$

$$\langle a_0, a_1, \ldots, a_{n-1}\rangle \alpha \langle b_0, b_1, \ldots, b_{n-1}\rangle \leftrightarrow a_0 - a_1 = b_0 - b_1$$

$$\langle a_0, a_1, \ldots, a_{n-1}\rangle \beta \langle b_0, b_1, \ldots, b_{n-1}\rangle \leftrightarrow a_0 + a_1 = b_0 + b_1$$

The congruences $\eta_0 \wedge \eta_1$, $\eta_0 \vee \eta_1$, η_0, η_1, and α form a sublattice $\mathbf{M}$ of Con$\mathbf{V}$ isomorphic to $\mathbf{M}_3$. The primitive positive

$$\Phi(x_0, x_1) \leftrightarrow \exists x_2, x_3\, (x_0\eta_0 x_2 \wedge x_2\eta_1 x_1 \wedge x_0\eta_1 x_3 \wedge x_3\eta_0 x_1 \wedge x_2\alpha x_3)$$

employing only η_0, η_1, and α defines β. But if $\mathcal{V}$ does not satisfy $x + x = 0$, then $\beta \notin \mathbf{M}$. Thus $\mathbf{M}$ is not closed under primitive positive definitions and is not a congruence lattice. Con$\mathbf{V}$ is not hereditary. □

Lemma 4.2. *Suppose that* $\mathbf{A}$ *is a finite congruence (power-)hereditary algebra which satisfies a nontrivial idempotent Mal'cev condition. If* Con$\mathbf{A}$ *has a sublattice* $\mathbf{M}$ *which is OPC, then there is a congruence (power-)hereditary vector space* $\mathbf{V}$ *of characteristic* 2 *so that* Con$\mathbf{V} \cong \mathbf{M}$.

Proof. Suppose that Con$\mathbf{A}$ contains an OPC sublattice $\mathbf{M}$ and that Con$\mathbf{A}$ is (power-)hereditary. If $\mathbf{M}$ has only two elements, this is trivial, so assume that $\mathbf{M}$ has more than two elements. Denote the bottom and top elements of $\mathbf{M}$ by 0_M and 1_M. $\mathbf{A}/0_M$ satisfies the idempotent Mal'cev condition, has a copy of $\mathbf{M}$ in its congruence lattice, and is congruence (power-)hereditary by 3.8, so we can assume that $0_M = 0_A$. Suppose that B is an equivalence class of 1_M. Then $\mathbf{A}|_B$ is congruence (power-)hereditary by 3.4. Also, since B will be closed under any idempotent operation of $\mathbf{A}$, $\mathbf{A}|_B$ will satisfy the same idempotent Mal'cev condition. Moreover, if $B_1, B_2, \ldots, B_n$ are all of the equivalence classes of 1_M, then the interval $[0_A, 1_M]$ in Con$\mathbf{A}$ can be embedded in $\prod_{i=1}^n \mathrm{Con}\mathbf{A}|_{B_i}$ via the map $\theta \to \langle \theta|_{B_1}, \theta|_{B_2}, \ldots, \theta|_{B_n}\rangle$ (as in the proof of Lemma 3.4 in [7]). This implies that for some i the restriction of the equivalence relations in $\mathbf{M}$ to B_i is a lattice injection (by the simplicity of $\mathbf{M}$). Since B_i is an equivalence class of 1_M, this injection must map 1_M to 1_{B_i}. Also, the identity relation 0_M must map to 0_{B_i}. Thus some $\mathbf{A}|_{B_i}$ has a copy of $\mathbf{M}$ in its congruence lattice whose least element is 0_{B_i} and whose greatest element is 1_{B_i}. Furthermore, $\mathbf{A}|_{B_i}$ is congruence (power-)hereditary and satisfies a nontrivial idempotent Mal'cev condition. Replace $\mathbf{A}$ with this $\mathbf{A}|_{B_i}$. Since Con$\mathbf{A}$ is hereditary, we can add operations to $\mathbf{A}$ so that Con$\mathbf{A} = \mathbf{M}$ without losing satisfaction of the nontrivial idempotent Mal'cev condition.

We have an algebra $\mathbf{A}$ with Con$\mathbf{A}$ (power-)hereditary and OPC and so that $\mathbf{A}$ satisfies a nontrivial idempotent Mal'cev condition. Con$\mathbf{A}$ is tight, so we can apply the tame congruence theory outlined in Section 3. Let U be a $\langle 0_A, 1_A\rangle$-minimal set. We know the following about $\mathbf{A}|_U$:

(1) $\mathbf{A}|_U$ is minimal by 3.6.
(2) Con$\mathbf{A}|_U \cong \mathbf{M}$ by 3.6 and 3.1 since Con$\mathbf{A}$ is simple.
(3) $\mathbf{A}|_U$ is not polynomially equivalent to a $\mathbf{G}$-set by 3.11. (Note that $\mathbf{A}|_U$ consists of a single trace since U is a $\langle 0_A, 1_A\rangle$-minimal set.)
(4) $\mathbf{A}|_U$ is congruence (power-)hereditary by 3.2 and 3.6.

Since $\mathbf{A}|_U$ is minimal, it is (polynomially equivalent to) one of the types of algebras listed in 3.10. Since $\mathrm{Con}\mathbf{A}|_U$ has more than two elements, $\mathbf{A}|_U$ must have more than two elements, so $\mathbf{A}|_U$ is either a $\mathbf{G}$-set or a vector space. We have noted that since $\mathbf{A}$ satisfies a nontrivial idempotent Mal'cev condition, $\mathbf{A}|_U$ cannot be a $\mathbf{G}$-set by 3.11. Thus $\mathbf{A}|_U$ must be a vector space. Since $\mathrm{Con}\mathbf{A}|_U$ is (power-)hereditary, $\mathbf{A}|_U$ as a vector space must be of characteristic two by 4.1. □

If $\mathbf{M}$ in this proof is isomorphic to $\mathbf{M}_4$, then the vector space has dimension two and characteristic two. Then, the vector space is isomorphic to $\mathbf{F}^2$ for some field $\mathbf{F}$ with 2^k elements for some k. However, the congruence lattice of this vector space is $\mathbf{M}_{2^k+1}$, so we would have a contradiction. Hence we have established the following:

Corollary 4.3. *Suppose that* $\mathbf{A}$ *is a finite algebra which satisfies a nontrivial idempotent Mal'cev condition. If* $\mathrm{Con}\mathbf{A}$ *has a sublattice isomorphic to* $\mathbf{M}_4$*, then* $\mathbf{A}$ *is not congruence hereditary.*

This fact will allow us to say what the vector spaces in Lemma 4.2 are:

Corollary 4.4. *Suppose that* $\mathbf{A}$ *is a finite congruence (power-)hereditary algebra which satisfies a nontrivial idempotent Mal'cev condition. If* $\mathrm{Con}\mathbf{A}$ *has a sublattice* $\mathbf{M}$ *which is OPC, then there is a congruence (power-)hereditary vector space* $\mathbf{V}$ *over* $\mathbb{F}_2$ *so that* $\mathrm{Con}\mathbf{V} \cong \mathbf{M}$.

Proof. There is a congruence (power-)hereditary vector space $\mathbf{V}$ of characteristic two with $\mathrm{Con}\mathbf{V} \cong \mathbf{M}$ by Lemma 4.2. If $\mathbf{V}$ is simple (so $|\mathbf{M}| = 2$), then the corollary is trivial. Suppose that $\mathbf{V}$ is not simple. For some finite field $\mathbf{F}$ and for some $k \geq 2$, $\mathbf{V}$ is isomorphic to $\mathbf{F}^k$ as an $\mathbf{F}$-vector space. We prove that $\mathbf{F}$ must be $\mathbb{F}_2$. By 4.1, we know that $\mathbf{F}$ must be of characteristic two. This implies that $|\mathbf{F}| = 2^n$ for some $n \geq 1$. Suppose by way of contradiction that $n > 1$. The $\mathrm{Con}\mathbf{F}^k$ contains a copy of $\mathrm{Con}\mathbf{F}^2$, which is isomorphic to $\mathbf{M}_{2^n+1}$. Since $n > 1$, this means that $\mathrm{Con}\mathbf{F}^k$ contains a copy of $\mathbf{M}_4$. By Corollary 4.3, this would mean that $\mathrm{Con}\mathbf{V}$ is not (power-)hereditary. This is a contradiction, so it must be that $n = 1$ and $|\mathbf{F}| = 2$. Thus, $\mathbf{F} \cong \mathbb{F}_2$. □

In [2], Hegedűs and Pálfy give the following complete characterization of congruence (power-)hereditary Abelian p-groups:

Theorem 4.5. (Theorem 5.8 of [2]) *Suppose that* $\mathbf{A}$ *is a finite Abelian* p*-group for some prime* p*.*

(1) $\mathrm{Con}\mathbf{A}$ *is hereditary if and only if either* $\mathbf{A}$ *is cyclic, or* $\mathbf{A} = \mathbb{Z}_2 \times \mathbb{Z}_{2^k}$ *for some* $k \geq 1$*, or* $\mathbf{A} = \mathbb{Z}_2^3$.
(2) $\mathrm{Con}\mathbf{A}$ *is power-hereditary if and only if either* $\mathbf{A}$ *is cyclic, or* $\mathbf{A} = \mathbb{Z}_2 \times \mathbb{Z}_{2^k}$ *for some* $k \geq 1$.

Suppose that $\mathbf{V}$ is a non-simple congruence (power-)hereditary vector space. Con$\mathbf{V}$ is OPC, so Corollary 4.4 tells us that there is a congruence (power-)hereditary vector space $\mathbf{U}$ over $\mathbb{F}_2$ with Con$\mathbf{U} \cong$ Con$\mathbf{V}$. This implies that $\mathbf{V} \cong \mathbf{U}$ (the congruence lattice of a non-simple finite vector space dictates the dimension and size of the vector space and hence determines the vector space up to isomorphism), so $\mathbf{V}$ is a vector space over $\mathbb{F}_2$. Now, the congruences of $\mathbf{V}$ as a vector space over $\mathbb{F}_2$ are identical to the congruences of the Abelian group reduct of $\mathbf{V}$. Hence by Theorem 4.5, $\mathbf{V}$ must be $\mathbb{Z}_2^2$ or $\mathbb{Z}_2^3$. On the other hand, if $\mathbf{V}$ is $\mathbb{Z}_2^2$ or $\mathbb{Z}_2^3$ treated as a vector space, then the congruences of $\mathbf{V}$ are identical to the congruences of the Abelian group reduct of $\mathbf{V}$. Hence we have the following:

Theorem 4.6. *Suppose that* $\mathbf{V}$ *is a finite vector space.*

(1) $\mathbf{V}$ *is congruence hereditary if and only if either* $\mathbf{V}$ *is simple or* $\mathbf{V}$ *is* $\mathbb{Z}_2^2$ *or* $\mathbb{Z}_2^3$.
(2) $\mathbf{V}$ *is congruence power-hereditary if and only if either* $\mathbf{V}$ *is simple or* $\mathbf{V}$ *is* $\mathbb{Z}_2^2$.

If Con$\mathbf{A}$ is (power-)hereditary and contains a sublattice $\mathbf{M}$ which is OPC, and if $\mathbf{A}$ satisfies a nontrivial idempotent Mal'cev condition, then by Corollary 4.4 $\mathbf{M}$ must be the congruence lattice of one of the vector spaces in Theorem 4.6. This immediately gives the next theorem. The critical lattice Con$(\mathbb{Z}_2^3)$ in the theorem has seven atoms and seven coatoms, and every height two interval in the lattice is isomorphic to $\mathbf{M}_3$.

Theorem 4.7. *Suppose that* $\mathbf{A}$ *is a finite algebra satisfying a nontrivial idempotent Mal'cev condition.*

(1) *If* Con$\mathbf{A}$ *contains a copy of an OPC lattice other than* $\mathbf{2}$, $\mathbf{M}_3$, *or* Con$(\mathbb{Z}_2^3)$, *then* Con$\mathbf{A}$ *is not hereditary.*
(2) *If* Con$\mathbf{A}$ *contains a copy of an OPC lattice other than* $\mathbf{2}$ *or* $\mathbf{M}_3$, *then* Con$\mathbf{A}$ *is not power-hereditary.*

5. Affine complete algebras

An algebra $\mathbf{A}$ is *affine complete* if every operation on the universe of $\mathbf{A}$ which preserves every congruence of $\mathbf{A}$ is a polynomial operation of $\mathbf{A}$.

Lemma 5.1. *Suppose that* $\mathbf{A}$ *is a finite affine complete algebra and* e *is an idempotent unary polynomial of* $\mathbf{A}$ *with* $U = e(A)$. *Then* $\mathbf{A}|_U$ *is also affine complete.*

Proof. Suppose that f is an n-ary operation on U which preserves the congruences of $\mathbf{A}|_U$. Define g on $\mathbf{A}$ by $g(x_1, \ldots, x_n) = f(e(x_1), \ldots, e(x_n))$. Then g preserves the congruences of $\mathbf{A}$ (so it is a polynomial of $\mathbf{A}$), and in U the equation $f = e \circ g$ holds. This makes f a polynomial operation of $\mathbf{A}|_U$. □

The following corollary implies that information about affine complete $\mathbf{G}$-sets might be useful in addressing the question of which $\mathbf{M}_n$ are congruence lattices of finite algebras.

Corollary 5.2. *Suppose that n is a positive integer so that $\mathbf{M}_n$ is not the congruence lattice of a finite vector space. If $\mathbf{M}_n$ is the congruence lattice of a finite algebra, then for some group $\mathbf{G}$, there is a finite affine complete $\mathbf{G}$-set whose congruence lattice is isomorphic to $\mathbf{M}_n$.*

Proof. Let $\mathbf{A}$ be a finite algebra with Con$\mathbf{A} \cong \mathbf{M}_n$. By adding the necessary operations to $\mathbf{A}$, we can assume that $\mathbf{A}$ is affine complete. Let U be any $\langle 0, 1\rangle$-minimal set of $\mathbf{A}$. The algebra $\mathbf{A}|_U$ is minimal and has a congruence lattice isomorphic to $\mathbf{M}_n$. It must be that $\mathbf{A}|_U$ is polynomially equivalent to a $\mathbf{G}$-set. By the the previous lemma, this $\mathbf{G}$-set is affine complete. □

Of course, $\mathbf{M}_n$ can be replaced here with any OPC lattice that is not the congruence lattice of a vector space. In [1], it was proven that there exists a finite algebra $\mathbf{A}$ with Con$\mathbf{A}$ distributive but so that there are no operations on the universe of $\mathbf{A}$ compatible with the congruences of $\mathbf{A}$ which satisfy Jónsson's equations for distributivity. Since the lattices $\mathbf{M}_n$ are modular, and since there are n for which $\mathbf{M}_n$ is representable as a congruence lattice but not as the congruence lattice of a vector space (such as $\mathbf{M}_7$), Corollary 5.2 gives the following:

Corollary 5.3. *There exists a finite algebra $\mathbf{A}$ with a modular congruence lattice so that there are no operations on the universe of $\mathbf{A}$ compatible with the congruences of $\mathbf{A}$ which satisfy Gumm's equations (or Day's equations) for congruence modularity.*

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