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Relations compatible with near unanimity operations

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Abstract. Given a system $\mathcal{R}$ of k-ary relations on a finite set A which are compatible with a $(k+1)$-ary near unanimity operation on A, we provide a characterization of when $\mathcal{R}$ is the system of all k-ary subuniverses of an algebra $\mathbf{A}$ on A.

1. Introduction

A near unanimity operation is an operation T which satisfies the equation

$$T(x, x, \ldots, x, y, x, \ldots, x, x) \approx x$$

for each location of the lone y. Baker and Pixley proved in [1] that if a variety $\mathcal{V}$ has a $(k+1)$-ary near unanimity term operation ($k \geq 2$) then any subalgebra of a product of finitely many algebras from $\mathcal{V}$ is determined uniquely by its projections to k coordinates. In [3], G. Bergman complemented the Baker-Pixley result by characterizing the system of k-ary projections of a subalgebra of a product of algebras in a variety with a $(k+1)$-ary near unanimity term operation. In the case where $k = 2$, this characterization is expressed in terms of relation composition, converse, and intersection. In this paper, we will define a k-ary composition operation on k-ary relations which we will use to give a similar charaterization for $k > 2$.

Using the characterizations in [3], C. Bergman [2] has shown that if a finite algebra $\mathbf{A}$ has a majority term operation (a ternary near unanimity operation) then $\mathbf{A}$ is determined up to categorical equivalence by the behavior of subuniverses of $\mathbf{A}^2$ under intersection, composition, and converse. Suppose that $\mathcal{R}$ is a set of subsets of A^2 for some finite set A so that $\mathcal{R}$ contains A^2 and the binary diagonal and is closed under intersection, composition, and converse. Further suppose that the sets in $\mathcal{R}$ are closed under a majority operation on A. C. Bergman asks in [2] if there is a algebra $\mathbf{A}$ on A so that $\mathcal{R}$ is the set of all subuniverses of $\mathbf{A}^2$. A positive answer to this question was found independently by this author, L. Zadori, and

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others. In this paper, we provide an extension of this to relations compatible with a near unanimity operation of any rank.

2. Compatible relations and functions

If A and X are sets, we will use A^X to represent the set of all functions from X to A. We will follow the tradition that if n is a positive integer, then $n = \{0, 1, \ldots, n-1\}$. With this convention, $x \in n$ will mean the same as $0 \leq x < n$. For any positive integer n, elements of A^n will sometimes be written as functions and other times as n-tuples. Hence, the i^{th} coordinate of $x \in A^n$ will sometimes be written as x_i and sometimes as $x(i)$.

If n is a positive integer, then an n-ary relation on a set A is a subset of A^n. If α is an n-ary relation on a set A and f is any operation on A, then α is compatible with f if α is closed under the induced operation f^{A^n}. If $\mathbf{A}$ is an algebra on A, then α is compatible with $\mathbf{A}$ if α is compatible with all basic operations of $\mathbf{A}$. If $\mathbf{A}$ is any algebra and n is a positive integer, let $\mathcal{R}_n(\mathbf{A})$ be the set of all n-ary compatible relations of $\mathbf{A}$. Let $\mathcal{R}(\mathbf{A}) = \bigcup_{n=1}^{\infty} \mathcal{R}_n(\mathbf{A})$.

Let A, X, and Y be any sets. Suppose $f\colon Y \to X$ is any function. Define $P_f\colon A^X \to A^Y$ by $P_f(g) = g \circ f$ for any $g \in A^X$. We will refer to $P_f(g)$ as a combination of the coordinates of g and will call P_f a coordinate combination function. If $k > 0$ and $J \subseteq k$ with $|J| = n$, then the projection of A^k to the coordinates in J is the function P_f where $f\colon n \to J$ is the unique increasing bijection. Coordinate combination functions will be pervasive throughout our work here. We will occasionally use the following properties (which the reader should verify).

Lemma 2.1. *Suppose that A, X, Y, and Z are sets and $f\colon X \to Y$ and $g\colon Y \to Z$.*

(1) *If f is injective, then P_f is surjective.*
(2) *If f is surjective, then P_f is injective.*
(3) *If f is bijective, then P_f is bijective and $P_f^{-1} = P_{f^{-1}}$.*
(4) *$P_f \circ P_g = P_{g \circ f}$.*
(5) *If $B \subseteq A^Z$, then $P_g^{-1}(P_f^{-1}(B)) = P_{g \circ f}^{-1}(B)$.*

First we note that coordinate combination functions can be used to characterize which systems of relations on a finite set A are of the form $\mathcal{R}(\mathbf{A})$ for some algebra $\mathbf{A}$ with universe A. There are a variety of similar characterizations. For example, such systems are characterized by being closed under logical definitions using primitive positive formulas [4, 6]. Also, these systems are exactly those which contain the diagonal relations and are closed under intersections, projections, and products of relations [4]. For similar references, the reader can see [5].

Theorem 2.2. *Suppose that $\mathcal{R}$ is a set of finitary relations on a finite set A. There is an algebra $\mathbf{A}$ on A with $\mathcal{R} = \mathcal{R}(\mathbf{A})$ if and only if these three conditions are met:*

(1) $A \in \mathcal{R}$.
(2) *$\mathcal{R}$ is closed under intersections of relations of the same rank.*
(3) *$\mathcal{R}$ is closed under P_f and P_f^{-1} for all positive integers n and m and all functions $f\colon n \to m$.*

3. Near Unanimity Terms

Suppose that $\mathcal{V}$ is a variety having a $(k+1)$-ary near unanimity term. Let $r > k$, and let $\mathbf{A}_0, \ldots, \mathbf{A}_{r-1} \in \mathcal{V}$. For each $I \subseteq r$ with $|I| = k$, let $\mathbf{S}_I$ be a subalgebra of $\prod_{i\in I} \mathbf{A}_i$. For any $J \subset r$ with $|J| > k$, if there is a subalgebra $\mathbf{S}$ of $\prod_{i\in J} \mathbf{A}_i$ so that the projection of $\mathbf{S}$ to the coordinates in I is precisely $\mathbf{S}_I$ for every $I \subseteq J$ with $|I| = k$, then we will say that the system $\{S_I : I \subseteq r \text{ and } |I| = k\}$ is **Bergman-consistent on** J. G. Bergman [3] gives the following characterization of Bergman-consistent systems. (Bergman calls these systems consistent. We call them Bergman-consistent to avoid confusion with our own notion of consistent below.)

Theorem 3.1 (G. Bergman [3])**.** *Let $k \geq 2$ and let $\mathcal{V}$ be a variety with a $(k+1)$-ary near unanimity term. Let $\mathbf{A}_0, \mathbf{A}_2, \ldots, \mathbf{A}_{r-1} \in \mathcal{V}$ (with $r > k$), and for every subset $I \subseteq r$ of cardinality k, let $\mathbf{S}_I$ be a subalgebra of $\prod_{i\in I} \mathbf{A}_i$. Then there exists a subalgebra of $\prod_{i\in r} \mathbf{A}_i$ whose projection to the coordinates in each I with $|I| = k$ is $\mathbf{S}_I$ (that is, the given system is Bergman-consistent on r) if and only if the system $\{S_I : I \subseteq r \text{ and } |I| = k\}$ is Bergman-consistent on every $J \subseteq r$ with $|J| = k+1$.*

Bergman's theorem has the following corollary when restriced to direct powers.

Corollary 3.2. *Suppose that $\mathbf{A}$ is an algebra with a $(k+1)$-ary near unanimity term for some $k \geq 2$. Let $m > k$ and for each $I \subseteq m$ with $|I| = k$ let $\mathbf{B}_I$ be a subalgebra of $\mathbf{A}^I$. There is subalgebra $\mathbf{C}$ of $\mathbf{A}^m$ so that for any $I \subseteq m$ with $|I| = k$ the projection of $\mathbf{C}$ to the coordinates in I is $\mathbf{B}_I$ if and only if the system $\{B_I : I \subseteq m \text{ and } |I| = k\}$ is Bergman-consistent on any $J \subseteq m$ with $|J| = k+1$.*

Suppose that $\mathbf{A}$ is an algebra with a ternary near unanimity term. In [3], Bergman gives a characterization in terms of relation composition and converse for a set of binary compatible relations on $\mathbf{A}$ to be the set of all binary projections of a subalgebra of $\mathbf{A}^X$. In this section, we develop a similar characterization using combinations of coordinates and a k-ary composition operation. We then use this to characterize when a set of k-ary relations on a finite set is the set of all k-ary compatible relations of an algebra with a $(k+1)$-ary near unanimity term.

Suppose $k \geq 2$. For any $i \in k+1$, let $\iota_i^k \colon k \to k+1$ be the unique increasing map whose range omits i. Usually, k will be understood, so we will write simply ι_i for ι_i^k.

Let A be any set. If $r_0, \ldots, r_{k-1} \subseteq A^k$, define

$$c_k(r_0, \ldots, r_{k-1}) = P_{\iota_k}\left(\bigcap_{i=0}^{k-1} P_{\iota_i}^{-1}(r_i)\right).$$

If r_i is assumed to be the projection of a $(k+1)$-ary relation B to all coordinates other than i for all $i = 0, \ldots, k-1$, then $c_k(r_0, \ldots, r_{k-1})$ is a candidate for the projection of B to the first k coordinates. Explicitly, $c_k(r_0, \ldots, r_{k-1})$ is the set

$$\Big\{\langle x_0, x_1, x_2, \ldots, x_{k-1}\rangle : (\exists x_k)\,(\langle x_1, x_2, x_3, \ldots, x_{k-1}, x_k\rangle \in r_0)\& \\ (\langle x_0, x_2, x_3, \ldots, x_{k-1}, x_k\rangle \in r_1)\& \\ (\langle x_0, x_1, x_3, \ldots, x_{k-1}, x_k\rangle \in r_2)\& \ldots \& \\ (\langle x_0, x_1, x_2, \ldots, x_{k-2}, x_k\rangle \in r_{k-1})\Big\}.$$

In particular, $c_2(r_0, r_1) = r_1 \circ r_0^{\cup}$. For any integers $k < m$, let $\mathrm{Inj}(k, m)$ be the set of all injective functions from k to m.

Definition 3.3. Let A be a set and let $m > k \geq 2$ be integers. Suppose that $\mathcal{R} = \{r_f : f \in \mathrm{Inj}(k, m)\}$ is a set of subsets of A^k. We will say that $\mathcal{R}$ is **(k, m)-consistent** if for all injective functions $f \colon k \to m$, $g \colon k \to k$, and $h \colon k+1 \to m$

$$P_g(r_f) = r_{f\circ g} \text{ and } r_{h\circ\iota_k} \subseteq c_k(r_{h\circ\iota_0}, r_{h\circ\iota_1}, \ldots, r_{h\circ\iota_{k-1}}).$$

Usually, k and m will be understood, and we will simply say that $\mathcal{R}$ is consistent. We will prove that in the presence of a $(k+1)$-ary near unanimity term $\mathcal{R} = \{r_f : f \in \mathrm{Inj}(k, m)\}$ is (k, m)-consistent if and only if there is an m-ary relation B on A so that $P_f(B) = r_f$ for all injective $f \colon k \to m$. We first approach $(k, (k+1))$-consistent systems.

Lemma 3.4. *Suppose that A is a set and $k \geq 2$. Let $\mathcal{R} = \{r_f : f \in \mathrm{Inj}(k, k+1)\}$ be a consistent set of k-ary relations on A. Define $B = \bigcap\{P_f^{-1}(r_f) : f \in \mathrm{Inj}(k, k+1)\}$. Then for any injective $f \colon k \to k+1$, $P_f(B) = r_f$.*

Proof. To begin with, define $C = \bigcap_{i=0}^{k} P_{\iota_i}^{-1}(r_{\iota_i})$. We will prove for all $j \in k+1$ that $P_{\iota_j}(C) = r_{\iota_j}$. Let $j \in k+1$, and let $g \colon k+1 \to k+1$ be given by

$$g(x) = \begin{cases} x & x < j \\ x+1 & j \leq x < k \\ j & x = k \end{cases}$$

Then $\iota_j = g \circ \iota_k$. Let $x \in r_{\iota_j}$. Then

$$x \in r_{\iota_j} = r_{g\circ\iota_k} \subseteq c_k(r_{g\circ\iota_0}, r_{g\circ\iota_1}, \ldots, r_{g\circ\iota_{k-1}}).$$

There is a $y \in A^{k+1}$ so that $P_{\iota_k}(y) = x$ and $P_{\iota_i}(y) \in r_{g\circ\iota_i}$ for $i = 0, 1, \ldots, k-1$. Let $z = y \circ g^{-1}$. Then $P_{g\circ\iota_i}(z) = P_{\iota_i}(y) \in r_{g\circ\iota_i}$ for all $i = 0, 1, \ldots, k$. Now, for each i, the range of $g \circ \iota_i$ is the same as the range of $\iota_{g(i)}$, so there is an injective function $h_i\colon k \to k$ with $g \circ \iota_i \circ h_i = \iota_{g(i)}$. We already know that $P_{g\circ\iota_i}(z) \in r_{g\circ\iota_i}$ for all i. Applying P_{h_i} to both sides of this inclusion along with consistency now gives $P_{\iota_{g(i)}}(z) \in r_{\iota_{g(i)}}$ for all i. This places $z \in C$. Since $P_{\iota_j}(z) = x$, we have that $r_{\iota_j} \subseteq P_{\iota_j}(C)$. The reverse inclusion is automatic from the definitions, so we have equality. Consistency now gives that $P_f(C) = r_f$ for any injective $f\colon k \to k+1$. This fact and the definitions of B and C imply that $B = C$. □

The following lemma is an easy extension of a result from [1].

Lemma 3.5 (Baker-Pixley [1])**.** *Suppose that* $\mathbf{A}$ *is an algebra with a* $(k+1)$*-ary near unanimity term for some* $k \geq 2$*. Let* $\mathbf{B}$ *and* $\mathbf{C}$ *be subalgebras of* $\mathbf{A}^m$ *for some* $m > k$*. Then* $B \subseteq C$ *if and only if* $P_f(B) \subseteq P_f(C)$ *for every injective* $f\colon k \to m$*.*

Lemma 3.6. *Let* $m > k \geq 2$ *be integers. Suppose that* $\mathbf{A}$ *is an algebra with a* $(k+1)$*-ary near unanimity term operation. Let* $\mathcal{R} = \{r_f : f \in \mathrm{Inj}(k,m)\}$ *be a consistent set of compatible* k*-ary relations on* $\mathbf{A}$*. Define* $B = \bigcap\{P_f^{-1}(r_f) : f \in \mathrm{Inj}(k,m)\}$*. Then for every injective* $f\colon k \to m$*,* $P_f(B) = r_f$*.*

Proof. For any $I \subset m$ with $|I| = k$, let $B_I = r_f$ where $f\colon k \to m$ is the unique increasing function with range I. We will prove that the system of B_I's is Bergman-consistent on m. To do so, it suffices to prove that for every $J \subset m$ with $|J| = k+1$, the system of B_I's with $I \subseteq J$ is Bergman-consistent.

Suppose that $J \subseteq m$ with $|J| = k+1$. Let $g\colon k+1 \to J$ be an increasing bijection. For each injective $f\colon k \to k+1$, let $s_f = r_{g\circ f}$. Then the family $\{s_f : f \in \mathrm{Inj}(k, k+1)\}$ inherits consistency from the set of r_f's. It follows from Lemma 3.4 that there is a subalgebra $\mathbf{B}$ of $\mathbf{A}^{k+1}$ which projects onto each s_f. This $\mathbf{B}$ corresponds naturally to a subalgebra of $\mathbf{A}^J$ which projects onto each $\mathbf{B}_I$ with $I \subset J$ and $|I| = k$. Thus, the system of B_I's is Bergman-consistent on every subset J of m with $k+1$ elements. By Theorem 3.2, it now follows that there is a subalgebra $\mathbf{C}$ of $\mathbf{A}^m$ whose projection to each $I \subset m$ with $|I| = k$ is B_I. Consistency now implies that $P_f(\mathbf{B}) = r_f$ for all injective $f\colon k \to m$. Lemma 3.5 implies that $\mathbf{C}$ must be the $\mathbf{B}$ defined in the statement of the theorem. □

It is simple to prove that if $B \subseteq A^m$ then the set $\{P_f(B) : f \in \mathrm{Inj}(k,m)\}$ is (k,m)-consistent. This and Lemmas 3.4, 3.5, and 3.6 now give us

Theorem 3.7. *Let $m > k \geq 2$ be integers. Suppose that* $\mathbf{A}$ *is an algebra with a $(k{+}1)$-ary near unanimity term operation. Let $\mathcal{R} = \{r_f : f \in \mathrm{Inj}(k,m)\}$ be a system of k-ary compatible relations on* $\mathbf{A}$*. $\mathcal{R}$ is (k,m)-consistent if and only if there is an m-ary compatible relation B on* $\mathbf{A}$ *so that $P_f(B) = r_f$ for all injective $f\colon : k \to m$. Furthermore, if such a B exists, then $B = \bigcap\{P_f^{-1}(r_f) : f \in \mathrm{Inj}(k,m)\}$.*

We are now ready to give our characterization of systems of k-ary relations which are precisely the systems of compatible k-ary relations on a finite algebra with a $(k+1)$-ary near unanimity operation.

Theorem 3.8. *Suppose that A is any finite set and that $k \geq 2$. Let $\mathcal{R}$ be a set of subsets of A^k so that*

(1) *$\mathcal{R}$ is closed under $\cap$, c_k, P_g and P_g^{-1} for all $g\colon k \to k$.*
(2) *$\mathcal{R}$ contains the k-ary diagonal subset and A^k.*

If $\mathcal{R}$ is compatible with a $(k+1)$-ary near unanimity operation T then there is an algebra $\mathbf{A}$ *on A with $\mathcal{R} = \mathcal{R}_k(\mathbf{A})$.*

Proof. Let $\mathcal{S}$ be the set of all relations on A of the form $\bigcap_{i=0}^{n} P_{f_i}^{-1}(r_i)$ where m is a positive integer, each $r_i \in \mathcal{R}$, and each $f_i\colon k \to m$. Each of these relations is a compatible relation on the algebra $\langle A, T\rangle$.

Claim If S is an m-ary member of $\mathcal{S}$ and $f\colon k \to m$, then $P_f(S) \in \mathcal{R}$.

Proof of Claim: Let $S \in \mathcal{S}$ be m-ary. There are $r_0, \dots r_n \in \mathcal{R}$ and $f_0, \dots, f_n\colon k \to m$ so that $S = \bigcap_{i=0}^{n} P_{f_i}^{-1}(r_i)$. Suppose first that $m \leq k$. Let $f\colon k \to m$. For each i, let $\bar{f}_i\colon k \to k$ be the composition of f_i followed by the inclusion of m in k. Define $\bar{f}$ similarly. Then $P_f(S) = P_{\bar{f}}\big(\bigcap_{i=0}^{n} P_{\bar{f}_i}^{-1}(r_i)\big)$, which is in $\mathcal{R}$ since $\mathcal{R}$ is closed under intersection and the operations $P_{\bar{f}}$ and $P_{\bar{f}_i}^{-1}$.

Next, suppose that $m > k$. For any $f\colon k \to m$ define $B_f = \{r \in \mathcal{R} : P_f(S) \subseteq r\}$. B_f is not empty since $A^k \in \mathcal{R}$. Also, let $q_f = \bigcap B_f$. Note that since $\mathcal{R}$ is closed under intersections, $q_f \in \mathcal{R}$, $P_f(S) \subseteq q_f$, and $q_f \in B_f$. We will prove that $P_f(S) = q_f$. This will place $P_f(S) \in \mathcal{R}$.

We first show that $S = \bigcap_{f\colon k \to m} P_f^{-1}(q_f)$. Since $P_f(S) \subseteq q_f$ for all $f\colon k \to m$, the forward inclusion is clear. From the definition of S, we know that $r_i \in B_{f_i}$ for each i, so $q_{f_i} \subseteq r_i$. Suppose that $x \in \bigcap_{f\colon k \to m} P_f^{-1}(q_f)$. Then for each i, $P_{f_i}(x) \in q_{f_i} \subseteq r_i$, so $x \in S$. Thus, $\bigcap_{f\colon k \to m} P_f^{-1}(q_f) \subseteq S$ also holds.

We next prove that $\{q_f : f \in \mathrm{Inj}(k,m)\}$ is consistent and that the equality $P_g(q_f) = q_{f \circ g}$ holds for **all** $g\colon k \to k$ and for **all** $f\colon k \to m$. It will follow then that $S = \bigcap\{P_f^{-1}(q_f) : f \in \mathrm{Inj}(k,m)\}$, and by Theorem 3.7 (since all of these relations are compatible with T) we will then have that $P_f(S) = q_f$ for any $f\colon k \to m$.

Let $g\colon k \to k$ and $f\colon k \to m$. We will show that $P_g(q_f) = q_{f \circ g}$. Let $r = P_g^{-1}(q_{f \circ g})$. We will establish that $P_g(r) = q_{f \circ g}$. Clearly $P_g(r) \subseteq q_{f \circ g}$ so we only need the reverse inclusion. Note that $P_{f \circ g}(S) \subseteq P_g(A^k)$ so that $P_g(A^k) \in B_{f \circ g}$.

This means that $q_{f\circ g} \subseteq P_g(A^k)$. It follows that $q_{f\circ g} \subseteq P_g(r)$. To see this, let $x \in q_{f\circ g}$. Since $q_{f\circ g} \subseteq P_g(A^k)$, there is some $y \in A^k$ with $P_g(y) = x$. Since $P_g(y) = x \in q_{f\circ g}$, we know that $y \in P_g^{-1}(q_{f\circ g}) = r$. This places $x \in P_g(r)$. We have established that $q_{f\circ g} \subseteq P_g(r)$ and hence that $P_g(r) = q_{f\circ g}$. We will now use this to show that $P_g(q_f) = q_{f\circ g}$. Let $x \in S$. Then $P_{f\circ g}(x) \in q_{f\circ g}$, so $P_f(x) \in P_g^{-1}(q_{f\circ g}) = r$. This places $r \in B_f$ so $q_f \subseteq r$, and hence $P_g(q_f) \subseteq P_g(r) = q_{f\circ g}$. Also, if $x \in S$, then $P_f(x) \in q_f$, so $P_{f\circ g}(x) = P_g(P_f(x)) \in P_g(q_f)$. It follows that $P_g(q_f) \in B_{f\circ g}$, so $q_{f\circ g} \subseteq P_g(q_f)$. This gives $q_{f\circ g} = P_g(q_f)$.

Suppose now that $g\colon k+1 \to m$ is any injective function. We must show that $q_{g\circ\iota_k} \subseteq c_k(q_{g\circ\iota_0}, \ldots, q_{g\circ\iota_{k-1}})$. To do so, we will show that $c_k(q_{g\circ\iota_0}, \ldots, q_{g\circ\iota_{k-1}}) \in B_{g\circ\iota_k}$. Let $x \in S$. Then we already know that $P_{g\circ\iota_i}(x) \in q_{g\circ\iota_i}$ for all i. This means that $P_{\iota_i}(P_g(x)) \in q_{g\circ\iota_i}$ for all i, so

$$P_{g\circ\iota_k}(x) = P_{\iota_k}(P_g(x)) \in c_k(q_{g\circ\iota_0}, \ldots, q_{g\circ\iota_{k-1}}).$$

Since this is true for all $x \in S$, it follows that $c_k(q_{g\circ\iota_0}, \ldots, q_{g\circ\iota_{k-1}}) \in B_{g\circ\iota_k}$ as desired. This completes the proof that $\{q_f : f \in \mathrm{Inj}(k, m)\}$ is consistent. Since the equality $P_g(q_f) = q_{f\circ g}$ holds for all $f\colon k \to m$ and $g\colon k \to k$, we can actually conclude that $S = \bigcap\{P_f^{-1}(q_f) : f \in \mathrm{Inj}(k, m)\}$. By consistency, it follows from Theorem 3.7 that for any injective $f\colon k \to m$, $P_f(S) = q_f \in \mathcal{R}$. If $f\colon k \to m$ is not injective, then we can factor f as $g \circ h$ for some $h\colon k \to k$ and some injective $g\colon k \to m$. Then

$$P_f(S) = P_{g\circ h}(S) = P_h(P_g(S)) = P_h(q_g) = q_{g\circ h} = q_f \in \mathcal{R}.$$

This ends the proof of the claim. One consequence of the claim is that the k-ary members of $\mathcal{S}$ are exactly the relations in $\mathcal{R}$, for if $S \in \mathcal{S}$ is k-ary and $f\colon k \to k$ is the identity function, then the claim tells us that $S = P_f(S)$ is in $\mathcal{S}$. We will use Theorem 2.2 to prove that there is an algebra $\mathbf{A}$ on A with $\mathcal{R}(\mathbf{A}) = \mathcal{S}$. First of all, $A \in \mathcal{S}$ because $A = P_f^{-1}(\delta_k)$ where δ_k is the k-ary diagonal and f is the constant function from k to 1. It should be obvious from the definition that $\mathcal{S}$ is closed under intersections.

Suppose that $B \in \mathcal{S}$ is m-ary and that $f\colon n \to m$ and $g\colon m \to l$. We will show that $P_f(B)$ and $P_g^{-1}(B)$ are in $\mathcal{S}$. There are $r_0, \ldots, r_t \in \mathcal{R}$ and $f_0, \ldots, f_t\colon k \to m$ so that $B = \bigcap_{i=0}^t P_{f_i}^{-1}(r_i)$. Notice that

$$\begin{aligned} P_g^{-1}(B) &= P_g^{-1}(\textstyle\bigcap_{i=0}^t P_{f_i}^{-1}(r_i)) \\ &= \textstyle\bigcap_{i=0}^t P_g^{-1}(P_{f_i}^{-1}(r_i)) \\ &= \textstyle\bigcap_{i=0}^t P_{g\circ f_i}^{-1}(r_i). \end{aligned}$$

Hence, $P_f^{-1}(B) \in \mathcal{S}$ by the definition of $\mathcal{S}$.

When considering $P_f(B)$, there are two cases: either $n > k$ or $n \le k$. Assume first that $n > k$. Since all of these relations are compatible with a $(k+1)$-ary near unanimity operation, Theorem 3.7 tells us that

$$P_f(B) = \bigcap_{h \in \mathrm{Inj}(k,n)} P_h^{-1}(P_h(P_f(B))) = \bigcap_{h \in \mathrm{Inj}(k,n)} P_h^{-1}(P_{f \circ h}(B)).$$

For any $h\colon k \to n$, $P_{f\circ h}(B)$ is in $\mathcal{R}$ by the Claim. Hence $P_f(B) \in \mathcal{S}$.

Next, assume that $n \le k$. Let $\iota\colon n \to k$ be the inclusion function, and let $f'\colon k \to n$ be any function so that $f = f' \circ \iota$. Then $P_f(B) = P_{f'\circ\iota}(B) = P_\iota(P_{f'}(B))$. It follows from the claim that $P_{f'}(B) \in \mathcal{R} \subseteq \mathcal{S}$. Let $D = P_{f'}(B)$. We need only show that $P_\iota(D) \in \mathcal{S}$. Let $h\colon k \to n$ be any function so that if $i < n$, then $h(i) = i$. We claim that $P_\iota(D) = P_h^{-1}(P_h(P_\iota(D)))$. The forward inclusion is obvious. Let $x \in P_h^{-1}(P_h(P_\iota(D)))$. Then $x \circ h \in P_h(P_\iota(D))$, so there is a $y \in P_\iota(D)$ with $x \circ h = y \circ h$. If $i < n$, then $x(i) = x \circ h(i) = y \circ h(i) = y(i)$. Since $x, y \in A^n$, this means that $x = y \in P_\iota(D)$. This establishes the desired equality. We have

$$P_\iota(D) = P_h^{-1}(P_h(P_\iota(D))) = P_h^{-1}(P_{\iota\circ h}(D)).$$

Since $D \in \mathcal{R}$ and $\mathcal{R}$ is closed under $P_{\iota\circ h}$, this places $P_\iota(D) \in \mathcal{S}$ as desired and completes the proof that $P_f(B) \in \mathcal{S}$.

The relations in $\mathcal{S}$ satisfy the conditions of Theorem 2.2 so there is an algebra $\mathbf{A}$ on A with $\mathcal{R}(\mathbf{A}) = \mathcal{S}$. From the Claim we know that the k-ary relations in $\mathcal{S}$ are precisely those in $\mathcal{R}$. The theorem now follows. □

Theorem 3.8 gives us this corollary for the special case when $k = 2$.

Corollary 3.9. *Suppose that $\mathcal{R}$ is a system of binary relations on a finite set A which is closed under composition, converse and intersection and which contains A^2 and the binary diagonal δ_A. If the relations in $\mathcal{R}$ are compatible with a majority operation on A, then there is an algebra $\mathbf{A}$ on A with $\mathcal{R} = \mathcal{R}_2(\mathbf{A})$.*

Proof. We apply Theorm 3.8 with $k = 2$. We already know that $\mathcal{R}$ is closed under intersection and contains A^2 and the binary diagonal. Also, if r and s are binary relations on A, then $c_2(r,s) = s \circ r^\cup$, so $\mathcal{R}$ is closed under c_2 also. We need only show that $\mathcal{R}$ is closed under P_g and P_g^{-1} for any $g\colon 2 \to 2$. Let $g\colon 2 \to 2$ and $r \in \mathcal{R}$. We proceed by cases on g. If g is the identity function, then $P_g(r) = P_g^{-1}(r) = r \in \mathcal{R}$. If g is the non-identity permutation, then $P_g(r) = P_g^{-1}(r) = r^\cup$, which is in $\mathcal{R}$. If g is the constant 0 function, then $P_g(r) = (r \circ r^\cup) \cap \delta_A \in \mathcal{R}$ and $P_g^{-1}(r) = (r \cap \delta_A) \circ A^2 \in \mathcal{R}$. The case when g is constantly 1 is similar.

We have shown that $\mathcal{R}$ satisfies the conditions of Theorem 3.8, so there is an algebra $\mathbf{A}$ on A with $\mathcal{R} = \mathcal{R}_2(\mathbf{A})$. □

4. Problems

We close with a few problems. For any algebra $\mathbf{A}$, let $\mathcal{S}_2(\mathbf{A})$ be the algebra $\langle \mathcal{R}_2(\mathbf{A}), A^2, \delta_A, \cap, \circ, \cdot^{\cup} \rangle$. According to [2], if $\mathbf{A}$ is a finite algebra with a majority operation, then $\mathbf{A}$ is determined up to categorical equivalence by the isomorphism class of $\mathcal{S}_2(\mathbf{A})$. Let $\mathcal{V}$ be the variety of algebras of the same type as $\mathcal{S}_2$-structures (two constants, two binary operations, a unary operation), and let $\mathcal{K}$ be the class of all algebras $\mathbf{R}$ in $\mathcal{V}$ for which there is a finite algebra $\mathbf{A}$ with a majority operation so that $\mathbf{R} \cong \mathcal{S}_2(\mathbf{A})$. Corollary 3.9 shows that $\mathcal{K}$ is closed under subalgebras. Moreover, suppose that $\mathbf{A}$ and $\mathbf{B}$ are finite algebras with majority term operations and that $\mathbf{C}$ is the nonindexed product of $\mathbf{A}$ and $\mathbf{B}$ (the universe of $\mathbf{C}$ is $A \times B$, and the operations of $\mathbf{C}$ are all operations T on $A \times B$ so that in the first coordinate T is a term operation of $\mathbf{A}$ and in the second T is a term operation of $\mathbf{B}$). Then $\mathcal{S}_2(\mathbf{C}) \cong \mathcal{S}_2(\mathbf{A}) \times \mathcal{S}_2(\mathbf{B})$. It follows that $\mathcal{K}$ is also closed under finite products. This motivates the following problems.

Problem 4.1. Is $\mathcal{K}$ closed under homomorphic images?

Problem 4.2. Find a set of quasi-identities Σ (or identites if the answer to 4.1 is yes) characterizing the quasivariety (variety) generated by $\mathcal{K}$.

More generally:

Problem 4.3. Which $\mathcal{S}_2$-structures are isomorphic to $\mathcal{S}_2(\mathbf{A})$ for a (finite) algebra $\mathbf{A}$ with a majority term operation (or for an $\mathbf{A}$ whose only basic operation is a majority operation).

Corollary 3.9 assumes that $\mathcal{R}$ is closed under a majority operation from the beginning. This leads naturally to:

Problem 4.4. Give conditions (preferably in terms of composition and converse) under which a set of binary relations on a finite set are compatible with a majority operation on the set.

Our next problem was posed by Kalle Kaarli. It essentially asks if the condition of finiteness is necessary in Corollary 3.9. Finiteness is imposed on us in Theorem 3.8 by our application of Theorem 2.2. Theorem 2.2 can easily be extended to infinite algebras; however, this infinite version requires the consideration of infinite direct powers. Theorem 3.2 can be extended to infinite algebras if closure under certain infinitary operations is assumed [3]. So an extension of Theorem 3.8 seems reasonable under these additional closure conditions. The author is interested in solving this problem without these additional assumptions.

Problem 4.5. Suppose that A is an infinite set and $\mathcal{R}$ is an algebraic closure system on A^2 closed under composition and converse and containing the binary diagonal. If the relations in $\mathcal{R}$ are compatible with a majority operation on A, is there an algebra $\mathbf{A}$ with universe A so that $\mathcal{R} = \mathcal{R}_2(\mathbf{A})$?

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