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Maltsev conditions and relations on algebras

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Abstract. We show that every sentence preserved by products in a purely relational first order language corresponds to a Maltsev condition on subalgebras of direct powers. Moreover, we establish that this correspondence captures all strong Maltsev conditions whose defining equations do not involve compositions. We then demonstrate how a broader version of the correspondence is sufficient to capture all Maltsev conditions when we restrict our attention to locally finite varieties.

1. Introduction

In 1954, A. I. Maltsev [10] proved that a variety has permuting congruences if and only if it has a ternary term p satisfying the identities

$$p(x, y, y) \approx x \text{ and } p(y, y, x) \approx x.$$

This was the first such discovered connection between a congruence property of a variety and the equations satisfied by the variety. In 1967, B. Jónsson [8] showed that congruence distributivity of a variety was also equivalent to the existence of particular terms of the variety satisfying certain equations. A similar characterization of congruence modularity was given by A. Day [2] in 1969. These discoveries have inspired a new way of classifying varieties according to properties of congruences on their algebras and according to existence conditions satisfied by their terms. They also have given powerful tools for the investigation of congruence distributive and congruence modular varieties. These “existence conditions” on terms are called Maltsev conditions. Since the discoveries of Maltsev, Jónsson, and Day, a number of Maltsev conditions have been found. While many of these involve congruences, some do not. For example the property that every two subalgebras of any algebra in a variety have non-empty intersection is equivalent to a Maltsev condition. Also, the requirement that a variety contains no two-element algebras is a Maltsev condition.

Suppose $\mathcal{L}$ is any purely relational first order language. We think of the relation symbols of $\mathcal{L}$ as being interpreted by compatible relations on algebras. We show that every sentence in $\mathcal{L}$ which is preserved by products corresponds naturally to a Maltsev condition equivalent

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to a condition on subalgebras of direct powers. Moreover, we also show that any strong Maltsev condition (to be defined) given by linear equations is equivalent in this manner to a sentence in some purely relational first order language.

2. Preliminaries

To begin with, we use the notion of interpretability to make the idea of a Maltsev condition rigorous. Suppose $\mathcal{V}$ is a variety with a set of basic operation symbols F, and suppose $\mathcal{W}$ is any variety. $\mathcal{W}$ is said to **interpret** $\mathcal{V}$ (or $\mathcal{V}$ is interpretable in $\mathcal{W}$-or there is an interpretation of $\mathcal{V}$ in $\mathcal{W}$) if and only if for every basic operation t of $\mathcal{V}$ there is a $\mathcal{W}$-term s_t so that for every algebra $\mathbf{A} \in \mathcal{W}$ the algebra $\langle A, \{s_t^A : t \in F\}\rangle$ is a member of $\mathcal{V}$. This relationship is denoted by $\mathcal{V} \leq \mathcal{W}$. It is equivalent to the condition that there be a mapping from Clo$\mathcal{V}$ to Clo$\mathcal{W}$ which preserves ranks and compositions.

We pause for a moment to address nullary operations. If a variety $\mathcal{V}$ has a nullary constant c, then it also has a unary constant $f(x) = c$ which mimics the nullary constant. Let $\mathcal{V}'$ be the variety with all but the nullary operations of $\mathcal{V}$. Then $\mathcal{V}' \leq \mathcal{V}$, but $\mathcal{V} \not\leq \mathcal{V}'$ since $\mathcal{V}'$ has no nullary operations. This distinction is somewhat confusing since $\mathcal{V}$ and $\mathcal{V}'$ are *essentially* the same. To avoid this confusion, we will follow [4] and assume that *algebras have no nullary operations*.

A class of $\mathcal{K}$ of varieties is a **strong Maltsev class** (or is defined by a **strong Maltsev condition**) if and only if there is a finitely presented variety $\mathcal{V}$ so that $\mathcal{K}$ is precisely the class of all varieties $\mathcal{W}$ for which $\mathcal{V} \leq \mathcal{W}$. If there is an infinite sequence of finitely presented varieties $\ldots \leq \mathcal{V}_3 \leq \mathcal{V}_2 \leq \mathcal{V}_1$ so that $\mathcal{K}$ is the class of all varieties $\mathcal{W}$ for which $\mathcal{V}_i \leq \mathcal{W}$ for some i, then $\mathcal{K}$ is a **Maltsev class** ($\mathcal{K}$ is defined by a **Maltsev condition**). Finally, if $\mathcal{K}$ is the intersection of countably many Maltsev classes, then $\mathcal{K}$ is a **weak Maltsev class** ($\mathcal{K}$ is defined by a **weak Maltsev condition**).

The **nonindexed product** of any collection $\{\mathbf{A}_i : i \in I\}$ of algebras (denoted $\otimes_{i\in I}\mathbf{A}_i$) is an algebra with universe $\prod_{i\in I} A_i$. If t_i is an n-ary term of $\mathbf{A}_i$ for each $i \in I$, then $\otimes_{i\in I}\mathbf{A}_i$ has an n-ary operation t given by $t(x_1, \ldots, x_n)(i) = t_i(x_1(i), \ldots, x_n(i))$. All such operations make up the basic operations of $\otimes_{i\in I}\mathbf{A}_i$. The nonindexed product of a collection of varieties $\{\mathcal{V}_i : i \in I\}$ is the variety generated by all $\otimes_{i\in I}\mathbf{A}_i$ where $\mathbf{A}_i \in \mathcal{V}_i$. It is easy to see that if $\mathbf{A}_1$ and $\mathbf{A}_2$ are algebras then any subalgebra of $\mathbf{A}_1 \otimes \mathbf{A}_2$ is the nonindexed product of a subalgebra of $\mathbf{A}_1$ and a subalgebra of $\mathbf{A}_2$. Similarly, any congruence of $\mathbf{A}_1 \otimes \mathbf{A}_2$ is the product of a congruence from $\mathbf{A}_1$ and one from $\mathbf{A}_2$. It is also not difficult to see that finite nonindexed products commute with direct products. From these facts, it follows that if $\mathcal{V}$ and $\mathcal{W}$ are varieties, then $\mathbf{A} \in \mathcal{V} \otimes \mathcal{W}$ if and only if there are algebras $\mathbf{B} \in \mathcal{V}$ and $\mathbf{C} \in \mathcal{W}$ so that $\mathbf{A} \cong \mathbf{B} \otimes \mathbf{C}$.

In [17], W. Taylor proves that a number of properties are equivalent to Maltsev conditions. His major tool is a characterization of Maltsev conditions equivalent to the following theorem. Our statement of the theorem is taken from [9].

THEOREM 2.1. *(W. Taylor* [17], *W. Neumann* [13]*) A class $\mathcal{K}$ of varieties is a Maltsev class if and only if the following hold:*

1. *Every variety in which some member of $\mathcal{K}$ has an interpretation belongs to $\mathcal{K}$.*
2. *$\mathcal{K}$ is closed under finite non-indexed products.*
3. *Every member of $\mathcal{K}$ is contained in a finitely based variety that also belongs to $\mathcal{K}$.*

3. Maltsev conditions and first order logic

Suppose $\mathcal{L}$ is a first order language with relation symbols $\{r_i : i \in I\}$ where each r_i has rank σ_i. If $\mathcal{V}$ is any variety, let $\mathcal{L}(\mathcal{V})$ be the class of all $\mathcal{L}$-structures $\langle A, \{r_i^A : i \in I\}\rangle$ so that there is an algebra $\mathbf{A}$ in $\mathcal{V}$ with universe A and r_i^A is a subuniverse of $\mathbf{A}^{\sigma_i}$ for each $i \in I$. If $\mathbf{A}$ is an algebra of type τ and r_i^A is a subuniverse of $\mathbf{A}^{\sigma_i}$ for each $i \in I$, we will call the structure $\langle \mathbf{A}, \{r_i^A : i \in I\}\rangle$ a $(\tau \cup \mathcal{L})$-structure. When there is no danger of confusion, we will omit superscripts on our relation and operation symbols.

We are ready for our first theorem:

THEOREM 3.1. *Suppose $\mathcal{L}$ is a purely relational first order language and σ is a sentence in $\mathcal{L}$ preserved by products. The class*

$$\{\mathcal{V} : \mathcal{L}(\mathcal{V}) \models \sigma\}$$

is a Maltsev class.

Proof. Since we are only concerned with the relation symbols in σ, we can assume $\mathcal{L}$ has finitely many relation symbols $r_1, \dots, r_n$ of rank $\sigma_1, \dots, \sigma_n$ respectively. Let $\mathcal{K}$ be the class of varieties defined in the theorem. We show that $\mathcal{K}$ satisfies the conditions of Theorem 2.1. That $\mathcal{K}$ satisfies the first condition is easy. The second condition follows quickly from the fact that for any varieties $\mathcal{V}$ and $\mathcal{W}$ every member of $\mathcal{L}(\mathcal{V} \otimes \mathcal{W})$ is isomorphic to a product of a member of $\mathcal{L}(\mathcal{V})$ and a member of $\mathcal{L}(\mathcal{W})$. The third condition requires a little more work.

Suppose that $\mathcal{V} \in \mathcal{K}$ is a variety of type τ. If t is a τ-term and r_i is a relation symbol from $\mathcal{L}$, then we can write a formula ϵ in the language of $(\tau \cup \mathcal{L})$-structures so that any $(\tau \cup \mathcal{L})$-structure $\langle \mathbf{A}, r_1^A, \dots, r_n^A\rangle$ models ϵ if and only if r_i^A is preserved by t^A. Let Δ be the collection of all such formulas for each τ-term t and each $i = 1, \dots, n$. Suppose that $\langle \mathbf{A}, r_1, \dots, r_n\rangle$ is a $(\tau \cup \mathcal{L})$-structure. If $\langle \mathbf{A}, r_1, \dots, r_n\rangle \models \mathrm{Id}(\mathcal{V}) \cup \Delta$ then $\mathbf{A} \in \mathcal{V}$ and $r_i \in \mathrm{Sub}\,\mathbf{A}^{\sigma_i}$ for each i. Therefore, the $\mathcal{L}$-structure $\langle A, r_1, \dots, r_n\rangle$ models σ, and hence the $(\tau \cup \mathcal{L})$-structure $\langle \mathbf{A}, r_1, \dots, r_n\rangle$ also models σ. Thus we see $\mathrm{Id}(\mathcal{V}) \cup \Delta \vdash \sigma$. We can find a finite subset $\Gamma \subseteq \mathrm{Id}(\mathcal{V})$ so that $\Gamma \cup \Delta \vdash \sigma$. Let $\mathcal{W}$ be the variety of type τ defined by Γ. $\mathcal{V}$ is contained in $\mathcal{W}$, and $\mathcal{W}$ is finitely based. It is also the case that $\mathcal{W} \in \mathcal{K}$. For if $\mathbf{A} \in \mathcal{W}$ and $r_i \in \mathrm{Sub}\,\mathbf{A}^{\sigma_i}$ for each i, then $\langle \mathbf{A}, r_1, \dots, r_n\rangle \models \Gamma \cup \Delta$ and so $\langle \mathbf{A}, r_1, \dots, r_n\rangle \models \sigma$. Hence $\langle A, r_1, \dots, r_n\rangle \models \sigma$. This shows that $\mathcal{K}$ satisfies (3) of Theorem 2.1 and completes the proof that $\mathcal{K}$ is a Maltsev class. ☐

We offer two applications of this theorem. Equations for the first example were given by W. Taylor in [16].

THEOREM 3.2. (*W. Taylor* [16]) *The class of all varieties which contain no two-element algebras is a Maltsev class.*

Proof. This class is defined in the manner of Theorem 3.1 by the following implication which is in every first order language and which is obviously preserved under products:

$$[\exists x_1, x_2(\neg(x_1 \approx x_2))] \longrightarrow [\exists x_1, x_2, x_3(\neg(x_1 \approx x_2) \bigwedge \neg(x_1 \approx x_3) \bigwedge \neg(x_2 \approx x_3))]$$

□

An algebra has **regular** congruences if and only if every congruence of the algebra is completely and uniquely determined by any equivalence class. A variety is regular if every member of the variety has regular congruences. In [19] H. A. Thurston proved that a variety $\mathcal{V}$ is regular if and only if the only congruence on any algebra in $\mathcal{V}$ which has a singleton congruence class is the identity relation. Using this fact it is simple to show:

THEOREM 3.3. (*R. Wille* [20], *G. Grätzer* [6], *B. Csákány* [*1*]) *The class of all varieties with regular congruences is a Maltsev class.*

Proof. Let $\mathcal{L}$ be the first order language with one binary relation symbol r, and let $\text{con}(r)$ be the sentence in $\mathcal{L}$ which holds if and only if r is an equivalence relation. The property "If r is an equivalence relation then it has a singleton class if and only if it is the identity relation" is preserved under products and is defined by this sentence in $\mathcal{L}$:

$$\text{con}(r) \longrightarrow [(\exists a \forall b(r(a, b) \longrightarrow (a \approx b))) \longrightarrow (\forall a, b(r(a, b) \longrightarrow (a \approx b)))]$$

□

4. Strong Maltsev classes

An obvious question to ask is which Maltsev classes can be described in this manner. We give a partial solution to this problem. We show that Theorem 3.1 is powerful enough to capture any strong Maltsev class corresponding to a variety which has a finite presentation consisting of only linear equations (i.e. equations not involving compositions). We call such Maltsev classes **linear (strong) Maltsev classes**.

THEOREM 4.1. *Every linear strong Maltsev class is defined by a sentence in a first order language with binary relation symbols of the form*

$$\forall \overline{x} \, \exists \overline{y} \left[\bigwedge_i \left[\left(\bigwedge_j r_i(a_j, b_j) \right) \longrightarrow r_i(c_i, d_i) \right] \right] \tag{1}$$

where all of the a's and b's are x's. Moreover, every sentence of this form defines a linear strong Maltsev class.

Proof. Suppose that a variety $\mathcal{V}$ has a finite linear presentation. We can assume for some n that $\mathcal{V}$ has a presentation $\langle f_1, \ldots, f_m, \sum\rangle$ in which every f_i is n-ary and every equation in $\sum$ involves only variables selected from $x_1, \ldots, x_n$. List the members of $\sum$ as:

$$\sum = \{f_{j_i}(x_{k_{i,1}}, \ldots, x_{k_{i,n}}) \approx f_{j'_i}(x_{k'_{i,1}}, \ldots, x_{k'_{i,n}}) : 1 \leq i \leq N\} \bigcup \{f_{h_i}(x_{l_{i,1}}, \ldots, x_{l_{i,n}}) \approx x_{l_{i,p_i}} : 1 \leq i \leq M\}.$$

Let $\mathcal{L}$ be the first order language with $2N + M$ binary relation symbols $r_1, \ldots, r_{2N+M}$, and let ϵ be this sentence in $\mathcal{L}$:

$$\forall x_1, \ldots, x_n \exists F_1, \ldots, F_m, y_1, \ldots, y_N$$

$$\bigwedge_{i=1}^{N} \left[\left(\left(\bigwedge_{t=1}^{n} r_i(x_{k_{i,t}}, x_t) \right) \longrightarrow r_i(y_i, F_{j_i}) \right) \wedge \tag{2}$$

$$\left(\left(\bigwedge_{t=1}^{n} r_{N+i}(x_{k'_{i,t}}, x_t) \right) \longrightarrow r_{N+i}(y_i, F_{j'_i}) \right) \right] \wedge \tag{3}$$

$$\bigwedge_{i=1}^{M} \left[\left(\bigwedge_{t=1}^{n} r_{2N+i}(x_{l_{i,t}}, x_t) \right) \longrightarrow r_{2N+i}(x_{l_{i,p_i}}, F_{h_i}) \right]. \tag{4}$$

We claim that a variety $\mathcal{W}$ interprets $\mathcal{V}$ if and only if $\mathcal{L}(\mathcal{W}) \models \epsilon$. Suppose first that $\mathcal{W}$ interprets $\mathcal{V}$. Then $\mathcal{W}$ has terms $f_1, \ldots, f_n$ modeling $\sum$. Suppose that $\mathbf{A} \in \mathcal{W}$ and $x_1, \ldots, x_n \in A$. For $i = 1, \ldots, n$, let $F_i = f_i^A(x_1, \ldots, x_n)$ and let

$$y_i = f_{j_i}^A(x_{k_{i,1}}, \ldots, x_{k_{i,n}}) = f_{j'_i}^A(x_{k'_{i,1}}, \ldots, x_{k'_{i,n}}).$$

If r is any subuniverse of $\mathbf{A}^2$ so that $\langle x_{k_{i,1}}, x_1\rangle, \ldots, \langle x_{k_{i,n}}, x_n\rangle \in r$, then

$$\langle y_i, F_{j_i}\rangle = f_{j_i}^{A^2}(\langle x_{k_{i,1}}, x_1\rangle, \ldots, \langle x_{k_{i,n}}, x_n\rangle) \in r.$$

Also, if $\langle x_{k'_{i,1}}, x_1\rangle, \ldots, \langle x_{k'_{i,n}}, x_n\rangle \in r$, then r also contains

$$\langle y_i, F_{j'_i}\rangle = f_{j'_i}^{A^2}(\langle x_{k'_{i,1}}, x_1\rangle, \ldots, \langle x_{k'_{i,n}}, x_n\rangle).$$

Thus the implications in (2) and (3) hold. The argument that (4) holds is identical, so we see that $\mathcal{L}(\mathcal{W}) \models \epsilon$.

Suppose that $\mathcal{L}(\mathcal{W}) \models \epsilon$. Let $\mathbf{A}$ be the free algebra in $\mathcal{W}$ generated by $\{x_1, \ldots, x_n\}$. Suppose $i \in \{1, \ldots, N\}$. Let r_i be the subuniverse of $\mathbf{A}^2$ generated by $\{\langle x_{k_{i,1}}, x_1\rangle, \ldots,$

$\langle x_{k_{i,n}}, x_n\rangle\}$ and let r_{N+i} be the subuniverse of $\mathbf{A}^2$ generated by $\{\langle x_{k'_{i,1}}, x_1\rangle, \ldots, \langle x_{k'_{i,n}}, x_n\rangle\}$. By ϵ, there are terms $f_1, \ldots, f_m$ of $\mathcal{W}$ so that $f_i^A(x_1, \ldots, x_n) = F_i$ for $i = 1, \ldots, m$ (The elements of $\mathbf{A}$ are precisely terms applied to $(x_1, \ldots, x_n)$). By (2) and (3), there is a y_i so that $\langle y_i, F_{j_i}\rangle \in r_i$ and $\langle y_i, F_{j'_i}\rangle \in r_{N+i}$, so there is a term g of $\mathcal{W}$ so that

$$g^A(x_1, \ldots, x_n) = F_{i_j} = F_{j_i}^A(x_1, \ldots, x_n) \textit{ and } g(x_{k_{i,1}}, \ldots, x_{k_{i,n}}) = y_i.$$

Since $\mathbf{A}$ is freely generated by $x_1, \ldots, x_n$ the variety $\mathcal{W}$ must satisfy $f_{j_i}(\bar{x}) \cong g(\bar{x})$. Thus $f_{j_i}^A(x_{k_{i,1}}, \ldots, x_{k_{i,n}}) = y_i$. Similarly we can show $f_{j'_i}^a(x_{k'_{i,1}}, \ldots, x_{k'_{i,n}}) = y_i$. Again, since $\mathbf{A}$ is freely generated by $x_1, \ldots, x_n$, it follows that $f_{j_i}(x_{k_{i,1}}, \ldots, x_{k_{i,n}}) \approx f_{j'_i}(x_{k'_{i,1}}, \ldots, x_{k'_{i,n}})$ holds in $\mathcal{W}$. A similar argument using (4) shows that $f_1, \ldots, f_m$ model all of the necessary identities of the form $f_{h_i}(x_{l_{i,1}}, \ldots, x_{l_{i,n}}) \approx x_{l_{i,p_i}}$. Thus $f_1, \ldots, f_m$ model $\sum$ in $\mathcal{W}$, so $\mathcal{V} \leq \mathcal{W}$.

We now show that every sentence of the form (1) defines a linear strong Maltsev class. Any sentence of this form can be expressed in this manner:

$$\begin{aligned}
&\forall x_1, \ldots, x_n \exists y_1, \ldots, y_m \\
&\quad \bigwedge_{i=1}^{N_1} \left[\left(\bigwedge_{j=1}^{M_i} r_i(x_{k_{i,j}}, x_{k'_{i,j}}) \right) \longrightarrow r_i(y_{a_i}, y_{a'_i}) \right] \\
&\quad \wedge \bigwedge_{i=N_1+1}^{N_2} \left[\left(\bigwedge_{j=1}^{M_i} r_i(x_{k_{i,j}}, x_{k'_{i,j}}) \right) \longrightarrow r_i(x_{b_i}, x_{b'_i}) \right] \\
&\quad \wedge \bigwedge_{i=N_2+1}^{N_3} \left[\left(\bigwedge_{j=1}^{M_i} r_i(x_{k_{i,j}}, x_{k'_{i,j}}) \right) \longrightarrow r_i(x_{c_i}, y_{c'_i}) \right] \\
&\quad \wedge \bigwedge_{i=N_3+1}^{N_4} \left[\left(\bigwedge_{j=1}^{M_i} r_i(x_{k_{i,j}}, x_{k'_{i,j}}) \right) \longrightarrow r_i(y_{d_i}, x_{d'_i}) \right].
\end{aligned}$$

Let $\mathcal{L}$ be the first order language with N_4 binary relation symbols $r_1, \ldots, r_{N_4}$, and let ϵ be this sentence in $\mathcal{L}$. For each $i = 1, \ldots, N_4$, let f_i be an M_i-ary operation symbol. Define

$$\begin{aligned}
\Gamma \;=\; &\{f_i(x_{k_{i,1}}, \ldots, x_{k_{i,M_i}}) \approx x_{b_i} : N_1 < i \leq N_2\} \\
&\bigcup \{f_i(x_{k'_{i,1}}, \ldots, x_{k'_{i,M_i}}) \approx x_{b'_i} : N_1 < i \leq N_2\} \\
&\bigcup \{f_i(x_{k_{i,1}}, \ldots, x_{k_{i,M_i}}) \approx x_{c_i} : N_2 < i \leq N_3\} \\
&\bigcup \{f_i(x_{k'_{i,1}}, \ldots, x_{k'_{i,M_i}}) \approx x_{d'_i} : N_3 < i \leq N_4\}
\end{aligned}$$

and

$$\begin{aligned}\Delta_1 = \quad & \{f_i(x_{k_{i,1}}, \ldots, x_{k_{i,M_i}}) \approx y_{a_i} : 1 \le i \le N_1\} \\ & \bigcup \{f_i(x_{k'_{i,1}}, \ldots, x_{k'_{i,M_i}}) \approx y_{a'_i} : 1 \le i \le N_1\} \\ & \bigcup \{f_i(x_{k'_{i,1}}, \ldots, x_{k'_{i,M_i}}) \approx y_{c'_i} : N_2 < i \le N_3\} \\ & \bigcup \{f_i(x_{k_{i,1}}, \ldots, x_{k_{i,M_i}}) \approx y_{d_i} : N_3 < i \le N_4\}.\end{aligned}$$

Next, let Δ be the collection of all identities $f_i(\overline{x}_1) \approx f_j(\overline{x}_2)$ where there is a y_l so that both $f_i(\overline{x}_1) \approx y_l$ and $f_i(\overline{x}_2) \approx y_l$ are in Δ_1. Finally, let $\mathcal{V}$ be the variety with presentation $\langle f_1, \ldots, f_{N_4}, \Gamma \cup \Delta\rangle$. A proof similar to the one for the first half of the theorem now shows that any variety $\mathcal{W}$ interprets $\mathcal{V}$ if and only if $\mathcal{L}(\mathcal{W}) \models \epsilon$. □

It should be noted that the variety $\mathcal{V}$ defined in the second half of this proof is often trivial. For example, if an identity

$$f_i(x_{k_{i,1}}, \ldots, x_{k_{i,M_i}}) \approx x_{b_i}$$

is included in Γ where x_{b_i} is not included in $\{x_{k_{i,1}}, \ldots, x_{k_{i,M_i}}\}$, then $\mathcal{V}$ is trivial.

5. Locally finite varieties

A variety $\mathcal{V}$ is **locally finite** if and only if for all $\mathbf{A} \in \mathcal{V}$ and for all $X \subseteq A$, if X is finite then so is the subalgebra it generates. In [7], the authors discuss Maltsev conditions restricted to the class of locally finite varieties. Throughout this sectioin, let $\mathcal{L}$ be the first order language with one relation symbol of every finite rank. We show that for any strong Maltsev class $\mathcal{K}$, there is a theory Γ in the language $\mathcal{L}$ so that if $\mathcal{V}$ is a locally finite variety, then $\mathcal{V} \in \mathcal{K}$ if and only if $\mathcal{L}(\mathcal{V}) \models \Gamma$.

For the sake of convenience, we will call a strong Maltsev class corresponding to a finitely presented variety defined using only n-ary operation symbols and only n variables an n-**ary strong Maltsev class**. It is not hard to see that every strong Maltsev class can be assumed to be n-ary for some n. We call the strong Maltsev condition associated with an n-ary strong Maltsev class an n-ary strong Maltsev condition. An n-ary strong Maltsev condition, then, is an assertion that there exist n-ary terms in a variety which satisfy certain equations in n variables.

LEMMA 5.1. *Suppose that Φ is an n-ary strong Maltsev condition and $m > 1$. Let $\mathcal{M}$ be the first order language with one m^n-ary relation symbol r. There is a sentence $\sigma_{\Phi,m}$ in $\mathcal{M}$ so that if $\mathbf{A}$ is any algebra, then $\langle A, r\rangle \models \sigma$ for all $r \in$ Sub $\mathbf{A}^{m^n}$ if and only if $\mathbf{A}$ has terms modeling Φ or $|\mathbf{A}| \neq m$.*

Proof. Suppose that Φ is an n-ary strong Maltsev condition. Let P_m be the sentence which holds in any $\mathcal{M}$-structure $\mathbf{M}$ if and only if $|\mathbf{M}| = m$. For $i = 1, \ldots, n$, let $\overline{p}_i \in \{x_1, \ldots, x_m\}^{m^n}$ be the projection to the i-th coordinate. Let F be the collection of all sets of n-ary operations on $\{x_1, \ldots, x_m\}$ satisfying the conditions of Φ, and let $\sigma_{\Phi,m}$ be this sentence in $\mathcal{L}$:

$$P_m \longrightarrow \left[\forall x_1, \ldots, x_m \left(\left[\bigwedge_{i \neq j \leq m} \neg(x_i \approx x_j)\right]\right.\right.$$
$$\left.\left.\longrightarrow \left[\left(\bigwedge_{i=1}^{n} r(\overline{p}_i)\right) \longrightarrow \bigvee_{T \in F} \left(\bigwedge_{\overline{f} \in T} r(\overline{f})\right)\right]\right)\right].$$

Suppose that $\mathbf{A}$ is an m element algebra and $\langle A, r\rangle \models \sigma_{\Phi,m}$ for all r in Sub $\mathbf{A}^{m^n}$. List the elements of $\mathbf{A}$ as $x_1, \ldots, x_m$, and let $r = \mathrm{Clo}_n\mathbf{A}$ (which is in Sub $\mathbf{A}^{m^n}$). Then $\neg(x_i \approx x_j)$ holds for each $i \neq j$ and $r(\overline{p}_i)$ holds for each $i = 1, \ldots, n$. From $\sigma_{\Phi,m}$ it follows that r must contain one of the sets of operations on A satisfying Φ. Thus $\mathbf{A}$ has terms modeling Φ.

On the other hand, suppose that $\mathbf{A}$ is an m element algebra with a set S of terms satisfying Φ. Suppose that $r \in$ Sub $\mathbf{A}^{m^n}$. We show that $\langle A, r\rangle \models \sigma_{\Phi,m}$. Let $\{x_1, \ldots, x_m\} \subseteq A$ be distinct. Then $\{x_1, \ldots, x_m\}$ is all of A and $S \in F$. If r contains all of the projection operations, then r contains $\mathrm{Clo}_n\mathbf{A}$ and hence contains all of S. Since $S \in F$, this shows $\langle A, r\rangle \models \sigma_{\Phi,m}$. □

For every m, the language $\mathcal{M}$ of this lemma is contained in $\mathcal{L}$, so we can assume that each $\sigma_{\Phi,m}$ is a sentence in $\mathcal{L}$. If $\mathcal{V}$ is locally finite and we insist that $\mathcal{L}(\mathcal{V})$ models each of $\sigma_{\Phi,2,}\sigma_{\Phi,3}, \ldots,$ then since $\mathbf{F}_{\mathcal{V}}(n)$ is finite, we know that this free algebra satisfies Φ. Hence, all of $\mathcal{V}$ must satisfy Φ. On the other hand, if a variety $\mathcal{V}$ satisfies Φ, then every finite algebra in the variety must satisfy Φ. Thus $\mathcal{L}(\mathcal{V})$ must satisfy all of the $\sigma_{\Phi,k}$. If we let $\Gamma = \sigma_{\Phi,2}, \sigma_{\Phi,3}, \ldots$ then we have:

THEOREM 5.2. *For any strong Maltsev condition Φ, there is a theory Γ in $\mathcal{L}$ so that any locally finite variety $\mathcal{V}$ satisfies Φ if and only if $\mathcal{L}(\mathcal{V}) \models \Gamma$.*

The language $\mathcal{L}$ is useful in describing all interpretations between locally finite varieties. The interpretability of one locally finite variety into another can always be described as a containment between classes of $\mathcal{L}$ structures.

LEMMA 5.3. *Suppose that $\mathcal{V}$ is a finitely presented variety and that $\mathcal{W}$ is locally finite. Then $\mathcal{V} \leq \mathcal{W}$ if and only if $\mathcal{L}(\mathcal{W}) \subseteq \mathcal{L}(\mathcal{V})$.*

Proof. The forward implication is obvious. Suppose that $\mathcal{L}(\mathcal{W})$ is contained in $\mathcal{L}(\mathcal{V})$. We can assume that $\mathcal{V}$ has a presentation $\langle \sum, F \rangle$ in which every operation symbol in F is n-ary and every equation in $\sum$ involves at most n variables. Let $\mathbf{A} = \mathbf{F}_{\mathcal{W}}(n)$. Since $\mathcal{W}$ is locally finite, $\mathbf{A}$ is finite. Let $m = |A|^n$ and consider $r = \mathrm{Clo}_n\mathbf{A} \in \mathrm{Sub}\ \mathbf{A}^m$. Since $\mathcal{L}(\mathcal{W}) \subseteq \mathcal{L}(\mathcal{V})$, we have that $\langle A, r \rangle \in \mathcal{L}(\mathcal{V})$, so there is an algebra $\mathbf{A}'$ on A in $\mathcal{V}$ so that $r \in \mathrm{Sub}\ (\mathbf{A}')^m$. Since r must contain the projection operations, $\mathrm{Clo}_n\mathbf{A}' \subseteq r$. Since $\mathrm{Clo}_n\mathbf{A}' \subseteq r = \mathrm{Clo}_n\mathbf{A}$, for each $f \in F$ there is a term $\hat{f}$ of $\mathcal{W}$ so that $f^{A'} = \hat{f}^A$. Then $\{\hat{f}^A : f \in F\}$ models $\sum$ in $\mathbf{A}$. Since $\mathbf{A} = \mathbf{F}_{\mathcal{W}}(n)$, these terms model $\sum$ on all of $\mathcal{W}$. This gives $\mathcal{V} \leq \mathcal{W}$ as desired. $\square$

We would like to eliminate the requirement that $\mathcal{V}$ be finitely presented in this lemma. In order to do this, we need the following lemma. A topological proof of this lemma is given in [12].

LEMMA 5.4. *For any varieties $\mathcal{V}$ and $\mathcal{W}$, if $\mathcal{W}$ is locally finite then $\mathcal{V} \leq \mathcal{W}$ if and only if $\mathcal{V}' \leq \mathcal{W}$ for all finitely presented $\mathcal{V}' \leq \mathcal{V}$.* $\square$

Using this we can show:

THEOREM 5.5. *Suppose that $\mathcal{V}$ and $\mathcal{W}$ are varieties and $\mathcal{W}$ is locally finite. Then $\mathcal{V} \leq \mathcal{W}$ if and only if $\mathcal{L}(\mathcal{W}) \subseteq \mathcal{L}(\mathcal{V})$.*

Proof. The forward implication is again trivial. Suppose that $\mathcal{L}(\mathcal{W}) \subseteq \mathcal{L}(\mathcal{V})$. If $\mathcal{V}' \leq \mathcal{V}$ is finitely presented, then $\mathcal{L}(\mathcal{W}) \subseteq \mathcal{L}(\mathcal{V}) \subseteq \mathcal{L}(\mathcal{V}')$. Since $\mathcal{V}'$ is finitely presented, Lemma 5.3, implies $\mathcal{V}' \leq \mathcal{W}$. Because this holds for all finitely presented $\mathcal{V}' \leq \mathcal{V}$, we have $\mathcal{V} \leq \mathcal{W}$ by the previous lemma. $\square$

6. Closing remarks

Theorem 3.1 allows us to look at the types of properties that are equivalent to Maltsev conditions in a clearer light. Essentially, anything that can be said about subuniverses of powers of algebras in a "nice" way corresponds to a Maltsev condition-where "nice" means first order and preserved by finite products. The ideas herein pose numerous questions. We state some of them here. We have begun to address the first problem-which (strong or weak) Maltsev conditions can be defined by sentences (or theories) in first order languages in the manner of Theorem 3.1? Some sentences in purely relational first order languages actually define strong Maltsev classes. Is there a method for determining whether a given sentence preserved by products defines a strong Maltsev class or a Maltsev class?

A. Pixley [14] and R. Wille [20] give algorithms for calculating the equations for (weak) Maltsev conditions equivalent to congruence identities. A similar algorithm for the types of sentences we are considering should prove quite useful.

In [3] it is shown that there are lattice identities which do not imply modularity but which when satisfied by all of the congruence lattices of a variety imply that every congruence lattice in the variety is modular. For sentences σ and ϵ in a purely relational first order language $\mathcal{L}$, it would be interesting to investigate the implication

$$(\mathcal{L}(\mathcal{V}) \models \sigma) \longrightarrow (\mathcal{L}(\mathcal{V}) \models \epsilon).$$

Finally, we would like to recall one of the supreme open problems in the area of Maltsev conditions - is every congruence identity (including joins) equivalent to a Maltsev condition (rather than merely a weak Maltsev condition)?

REFERENCES

[1] CSÁKÁNY, B., *Characterization of regular varieties*, Acta. Sci. Math. (Szeged), *31* (1970), 187–189.
[2] DAY, A., *A characterization of modularity for congruence lattices of algebras*, Canad. Math. Bull. *12* (1969), 167–173.
[3] DAY, A. and FREESE, R., *A characterization of identities implying congruence modularity*, Can. J. Math. *32* (1980), 1140–1167.
[4] GARCIA, O. C. and TAYLOR, W., *The lattice of interpretability types of varieties*, Amer. Math. Soc. Mem., 1984.
[5] GRÄTZER, G., *Universal Algebra*, The University Series in Higher Mathematics. D. Van Nostrand Co., Princeton, New Jersey, 1968.
[6] GRÄTZER, G., *Two mal'cev type theorems in universal algebra*, J. Combinatorial Theory, *8* (1970), 334–342.
[7] HOBBY, D. and MCKENZIE, R., *The Structure of Finite Algebras (tame congruence theory)*, Contemporary Mathematics. American Mathematical Society, Providence, RI, 1988.
[8] JÓNSSON, B., *Algebras whose congruence lattices are distributive*, Math. Scand. *21* (1967), 110–121.
[9] JÓNSSON, B., *Congruence varieties*, Algebra univers. *10* (1980), 355–394.
[10] MALTSEV, A. I., *On the general theory of algebraic systems*. Mat. Sbornik *77* (1954), 3–20.
[11] MCKENZIE, R., MCNULTY, G. and TAYLOR, W., *Algebras, Lattices, Varieties*, Volume I. Wadsworth and Brooks/Cole, Monterey, California, 1987.
[12] MCKENZIE, R. and ŚWIERCZKOWSKI, S., *Non-covering in the interpretability lattice of equational theories*, Algebra univers. *30* (1993), 157–170.
[13] NEUMANN, W. D., *On mal'cev conditions*, J. Austral. Math. Soc. *17* (1974), 376–384.
[14] PIXLEY, A. F., *Local malcev conditions*, Canad. Math. Bull. *15* (1972), 559–568.
[15] SNOW, J. W., *Relations on algebras and varieties as categories*, Dissertation, Vanderbilt University, 1998.
[16] TAYLOR, W., *Varieties without doubleton algebras*, Notices Amer. Math. Soc. *19* (1972), 753.
[17] TAYLOR, W., *Characterizing mal'cev conditions*, Algebra univers. *3* (1973), 351–397.

[18] TAYLOR, W., *Varieties obeying homotopy laws*, Canad. J. Math. *29* (1977), 498–527.
[19] THURSTON, H. A., *Derived operations and congruences*, Proc. London Math. Soc. *8* (1958), 127–134.
[20] WILLE, R., *Kongruenzklassengeometrien*, Springer-Verlag, New York, 1970. Lecture Notes in Mathematics, vol. 113.

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