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Every finite lattice in $\mathcal{V}(\mathrm{M}_3)$ is representable

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Abstract. We prove that every finite lattice in the variety generated by $\mathbf{M}_3$ is isomorphic to the congruence lattice of a finite algebra.

1. Introduction

Call a finite lattice which is isomorphic to the congruence lattice of a finite algebra **representable**. In [5], the author investigates constructions by which one can create new representable lattices from lattices already known to be representable. In this paper, we use similar tactics to prove that any finite lattice in the variety generated by $\mathbf{M}_3$ is representable.

2. A note on finitely fermentable lattices

In [3], Pudlák and Tůma introduced the idea of "finitely fermentable" lattices. A lattice $\mathbf{L}$ is finitely fermentable if and only if $\mathbf{L}$ is finite and $\mathbf{L}$ and Con$\mathbf{L}$ have the same number of join-irreducible elements. This is a property which is preserved under homomorphisms, sublattices, and finite direct products. Suppose that $\mathbf{L}$ is a finitely fermentable lattice. Every finite lattice in the variety generated by $\mathbf{L}$ is a homomorphic image of a sublattice of a finite direct power of $\mathbf{L}$. It follows that every finite lattice in the variety is also finitely fermentable. It is proven in [3] that every finitely fermentable lattice is representable. Hence, if $\mathbf{L}$ is a finitely fermentable lattice, every finite lattice in the variety generated by $\mathbf{L}$ is representable.

The lattice $\mathbf{M}_3$ is not finitely fermentable (it has four join irreducibles, but is simple), so the results of [3] do not apply to $\mathbf{M}_3$. However, we do note that two interesting lattices which are finitely fermentable are the two element lattice and the pentagon $\mathbf{N}_5$. It follows that every finite distributive lattice is representable (a well known fact [4]) and that every finite lattice in the variety generated by $\mathbf{N}_5$ is representable.

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3. Notation

To avoid confusion between logical symbols and lattice operations, we will use $\vee$ and $\wedge$ for logical "or" and "and." We will use additive notation ($+$, Σ) for lattice joins, and multiplicative notation ($\cdot$, Π, or juxtaposition) for lattice meets. We will refer to the intersection of arbitrary binary relations with $\cap$, while we refer to the intersection of equivalence relations with product notation. When equivalence relations might be mixed with non-equivalence relations, we will use $\cap$.

A primitive positive formula is a formula of the form $\exists \wedge$ (atomic). Suppose that ϕ is a primitive positive formula with two free variables v_1 and v_2 which employs binary relation symbols $r_1, \ldots, r_l$. Suppose also that $\theta_1, \ldots, \theta_l$ are binary relations on a set A. By interpreting each r_i as θ_i, Φ can be used to define a new binary relation on A. We will denote this new binary relation as $\Phi(\theta_1, \ldots, \theta_l)$. To denote that $\langle a, b\rangle$ satisfies this new relation, we will write $\langle a, b\rangle \in \Phi(\theta_1, \ldots, \theta_l)$ rather than $\Phi(\theta_1, \ldots, \theta_l)(a, b)$. On any set A, Φ induces an operation on binary relations of A which maps an l-tuple $\langle\theta_1, \ldots, \theta_l\rangle$ of binary relations to the relation $\Phi(\theta_1, \ldots, \theta_l)$. We will also call this operation Φ and usually denote it by $\Phi(r_1, \ldots, r_l)$. When there is possible confusion about the underlying set A involved, we will denote the operation by $\Phi^A(r_1, \ldots, r_l)$. We will also use $\Phi(r_1, \ldots, r_l)$ to denote the primitive positive formula. Generally, it will be apparent from context whether $\Phi(r_1, \ldots, r_l)$ refers to the formula or the operation.

By a compatible relation on an algebra $\mathbf{A}$ we mean a subuniverse of a (finite) direct power of A. In [5], the author exploits the following lemma which follows from the fact that a set of relations on a finite set is the set of all relations compatible with an algebra on the set if and only if the relations are closed under primitive positive definitions [1, 2].

Lemma 3.1 (Corollary 2.2 of [5])**.** *Suppose $\mathcal{L}$ is a 0–1 lattice of equivalence relations on a finite set A. There is an algebra $\mathbf{A}$ on A with $\mathrm{Con}\mathbf{A} = \mathcal{L}$ if and only if every equivalence relation on A which can be defined from $\mathcal{L}$ by a primitive positive formula is already in $\mathcal{L}$.*

4. The variety generated by $\mathbf{M}_3$

In this section, we prove

Theorem 4.1. *Every finite lattice in the variety generated by $\mathbf{M}_3$ is representable.*

We do so through the following sequence of definitions, lemmas, and corollaries. Let $B = \{1, 2, 3\}$. Denote the lattice of equivalence relations on B by $\mathrm{Eq}B$. Then $\mathrm{Eq}B$ is isomorphic to $\mathbf{M}_3$. Let $i \in B$. Denote the unique atom of $\mathrm{Eq}B$ in which

$\{i\}$ is an equivalence class by μ_i. Denote the universal relation on B by 1_B and the smallest equivalence relation by 0_B.

Lemma 4.2. *Suppose that α, β, and γ are equivalence relations on the three element set B. The equivalence relation $(\alpha+\beta)\gamma$ is the smallest equivalence relation on B containing $(\alpha\circ\beta)\cap\gamma$.*

Proof. Suppose that σ is an equivalence relation on B containing $(\alpha\circ\beta)\cap\gamma$. We will prove that $(\alpha+\beta)\gamma\leq\sigma$. If α or β is in $\{0_B,1_B\}$, then this is easy. Assume this is not the case. Then $\alpha=\mu_i$ and $\beta=\mu_j$ for some $i,j\in B$. If $i=j$ then

$$(\alpha+\beta)\gamma=\alpha\cap\gamma=(\alpha\circ\beta)\cap\gamma\leq\sigma.$$

Assume that $i\neq j$. We proceed now by cases on γ. If $\gamma=0_B$ then

$$(\alpha+\beta)\gamma=0_B\leq\sigma.$$

If $\gamma=1_B$, then $(\alpha\circ\beta)\cap\gamma=\alpha\circ\beta$. Since $(\alpha\circ\beta)\cap\gamma\leq\sigma$, and since $(\alpha\circ\beta)\cap\gamma=\alpha\circ\beta$, we know that $\alpha\leq\sigma$ and that $\beta\leq\sigma$. It follows then that

$$(\alpha+\beta)\gamma=\alpha+\beta\leq\sigma.$$

Next assume that $\gamma=\mu_k$ for some $k\in B$. If $k\in\{i,j\}$, then $(\alpha\circ\beta)\cap\gamma=\gamma$. If $k\notin\{i,j\}$, then $\langle j,i\rangle\in(\alpha\circ\beta)\cap\gamma$ while $\langle i,j\rangle\notin(\alpha\circ\beta)\cap\gamma$, so $0_B<(\alpha\circ\beta)\cap\gamma<\gamma$. In either case, $0_B<(\alpha\circ\beta)\cap\gamma\leq\gamma=\mu_k$, so the only equivalence relations above $(\alpha\circ\beta)\cap\gamma$ are $\mu_k=\gamma$ and 1_B. Therefore, either $\sigma=\gamma$ or $\sigma=1_B$. In either case, $(\alpha+\beta)\gamma\leq\sigma$. We have proven that if σ is any equivalence relation on B containing $(\alpha\circ\beta)\cap\gamma$ then $(\alpha+\beta)\gamma\leq\sigma$. This establishes the lemma since clearly $(\alpha\circ\beta)\cap\gamma\subseteq(\alpha+\beta)\gamma$. □

Definition 4.3. For any primitive positive formula $\Phi(r_1,\ldots,r_l)$ of the form

$$\langle v_1,v_2\rangle\in\Phi(r_1,\ldots,r_l)\leftrightarrow\exists v_3\bigwedge_{i=1}^{l}r_i(v_{j_i},v_{k_i})$$

define a lattice term $T_\Phi(r_1,\ldots,r_l)$ by

$$T_\Phi(r_1,\ldots,r_l)=\left(\left[\prod\{r_i:\{j_i,k_i\}=\{1,3\}\}\right]+\left[\prod\{r_i:\{j_i,k_i\}=\{3,2\}\}\right]\right)$$
$$\cdot\left[\prod\{r_i:\{j_i,k_i\}=\{1,2\}\}\right].$$

If any of the sets in this definition are empty, we follow the tradition that the meet over an empty set is the largest element of a lattice, so that this is actually a term in the language of lattices with a greatest element (the language with symbols $+$, $\cdot$, and 1).

Lemma 4.4. *If $\Phi(r_1, \dots, r_l)$ is a primitive positive formula of the form*

$$\langle v_1, v_2\rangle \in \Phi(r_1, \dots, r_l) \leftrightarrow \exists v_3 \bigwedge_{i=1}^{l} r_i(v_{j_i}, v_{k_i})$$

and if $\theta_1, \dots, \theta_l \in \mathrm{Eq}B$, then $T_\Phi(\theta_1, \dots, \theta_l)$ is the smallest equivalence relation on B containing $\Phi(\theta_1, \dots, \theta_l)$.

Proof. Define $\alpha = \prod\{\theta_i : \{j_i, k_i\} = \{1, 3\}\}$, $\beta = \prod\{\theta_i : \{j_i, k_i\} = \{3, 2\}\}$, and $\gamma = \prod\{\theta_i : \{j_i, k_i\} = \{1, 2\}\}$. Notice that each of these is an equivalence relation on B, and that these are the meets which occur within $T_\Phi(\theta_1, \dots, \theta_l)$ so that $T_\Phi(\theta_1, \dots, \theta_l) = (\alpha + \beta)\gamma$.

We claim that $\Phi(\theta_1, \dots, \theta_l) = (\alpha\circ\beta)\cap\gamma$. To see this, suppose first that $(x_1, x_2) \in \Phi(\theta_1, \dots, \theta_l)$. This means that there is an element $x_3 \in B$ so that $\theta_i(x_{j_i}, x_{k_i})$ is true for $i = 1, \dots, l$. Suppose that $\{j_i, k_i\} = \{1, 3\}$. Then from $\theta_i(x_{j_i}, x_{j_k})$, we know either $\theta_i(x_1, x_3)$ or $\theta_i(x_3, x_1)$. Since θ_i is symmetric, this means $\theta_i(x_1, x_3)$. This is true for all i for which $\{j_i, k_i\} = \{1, 3\}$, so $\langle x_1, x_3\rangle \in \alpha$. Similar arguments when $\{j_i, k_i\} = \{3, 2\}$ and $\{j_i, k_i\} = \{1, 2\}$ will establish that $\langle x_3, x_2\rangle \in \beta$ and $\langle x_1, x_2\rangle \in \gamma$. Hence $\langle x_1, x_2\rangle \in (\alpha \circ \beta) \cap \gamma$ and $\Phi(\theta_1, \dots, \theta_l) \subseteq (\alpha \circ \beta) \cap \gamma$.

Next, suppose that $\langle x_1, x_2\rangle \in (\alpha \circ \beta) \cap \gamma$. Since $\langle x_1, x_2\rangle \in \alpha \circ \beta$, there is an $x_3 \in B$ with $x_1\alpha x_3\beta x_2$. Suppose $i \in \{1, \dots, l\}$. If $\{j_i, k_i\} = \{1, 3\}$ then $\theta_i(x_1, x_3)$ (since $\langle x_1, x_3\rangle \in \alpha$). Since θ_i is symmetric, it follows that $\theta_i(x_{j_i}, x_{k_i})$. Similar arguments establish $\theta_i(x_{j_i}, x_{k_i})$ when $\{j_i, k_i\} = \{3, 2\}$ or $\{j_i, k_i\} = \{1, 2\}$. Thus we have $\theta_i(x_{j_i}, x_{k_i})$ for all i, so x_3 witnesses that $\langle x_1, x_2\rangle \in \Phi(\theta_1, \dots, \theta_l)$. This provides the reverse inclusion to conclude that $\Phi(\theta_1, \dots, \theta_l) = (\alpha \circ \beta) \cap \gamma$. The lemma now follows from Lemma 4.2. □

Definition 4.5. Suppose that $n \geq 2$. Define F_n to be the set of all functions $f : \{1, \dots, n\} \to \{1, 2, 3\}$ with $f(1) = 1$ and $f(2) = 2$. Suppose that $\Phi(r_1, \dots, r_l)$ is a primitive positive formula of the form

$$\langle u_1, u_2\rangle \in \Phi(r_1, \dots, r_l) \leftrightarrow \exists u_3, \dots, u_n \bigwedge_{i=1}^{l} r_i(u_{g_i}, u_{h_i}).$$

Let $f \in F_n$. Define $\Phi_f(r_1, \dots, r_l)$ to be the primitive positive formula

$$\langle v_1, v_2\rangle \in \Phi_f(r_1, \dots, r_l) \leftrightarrow \exists v_3 \bigwedge_{i=1}^{l} r_i(v_{f(g_i)}, v_{f(h_i)}).$$

Lemma 4.6. *Suppose that $\Phi(r_1, \dots, r_l)$ is the primitive positive formula*

$$\langle u_1, u_2\rangle \in \Phi(r_1, \dots, r_l) \leftrightarrow \exists u_3, \dots, u_n \bigwedge_{i=1}^{l} r_i(u_{g_i}, u_{h_i}).$$

Let $x_1, x_2 \in B$ and $\theta_1, \ldots, \theta_l \in \mathrm{Eq}B$. Then $\langle x_1, x_2\rangle \in \Phi(\theta_1, \ldots, \theta_l)$ if and only if there is a function $f \in F_n$ so that $\langle x_1, x_2\rangle \in \Phi_f(\theta_1, \ldots, \theta_l)$.

Proof. Suppose that $\langle x_1, x_2\rangle \in \Phi(\theta_1, \ldots, \theta_l)$. If $x_1 = x_2$, then $\langle x_1, x_2\rangle$ is in any $\Phi_f(\theta_1, \ldots, \theta_l)$. Suppose then that $x_1 \neq x_2$. There is exactly one element in B and not in $\{x_1, x_2\}$. Call this element z. There are $x_3, \ldots, x_n \in B$ so that $\theta_i(x_{g_i}, x_{h_i})$ for each $i = 1, \ldots, l$. Define $f : \{1, \ldots, n\} \to \{1, 2, 3\}$ by

$$f(i) = \begin{cases} 1 & x_i = x_1 \\ 2 & x_i = x_2 \\ 3 & x_i = z\ . \end{cases}$$

Let $y_1 = x_1$, $y_2 = x_2$, and $y_3 = z$. It follows that $y_{f(g_i)} = x_{g_i}$ and $y_{f(h_i)} = x_{h_i}$ for all i. Since $\theta_i(x_{g_i}, x_{h_i})$ for all i, this means $\theta_i(y_{f(g_i)}, y_{f(h_i)})$. Hence, $\langle x_1, x_2\rangle = \langle y_1, y_2\rangle \in \Phi_f(\theta_1, \ldots, \theta_l)$.

Next, suppose that $f \in F_n$ and that $\langle x_1, x_2\rangle \in \Phi_f(\theta_1, \ldots, \theta_l)$. This means there is an $x_3 \in B$ with $\theta_i(x_{f(g_i)}, x_{f(h_i)})$ for all i. For each $i = 1, \ldots, l$, let $y_i = x_{f(i)}$. Then $y_1 = x_1$, $y_2 = x_2$, and for all $i = 1, \ldots, l$, $y_{g_i} = x_{f(g_i)}$ and $y_{h_i} = x_{f(h_i)}$. Suppose that $i \in \{1, \ldots, l\}$. Since $\theta_i(x_{f(g_i)}, x_{f(h_i)})$, it follows that $\theta_i(y_{g_i}, y_{h_i})$. Thus $\langle x_1, x_2\rangle = \langle y_1, y_2\rangle \in \Phi(\theta_1, \ldots, \theta_l)$. □

Lemma 4.6 immediately gives us

Corollary 4.7. *Suppose that $\Phi(r_1, \ldots, r_l)$ is the primitive positive formula*

$$\langle u_1, u_2\rangle \in \Phi(r_1, \ldots, r_l) \leftrightarrow \exists u_3, \ldots, u_n \bigwedge_{i=1}^{l} r_i(u_{g_i}, u_{h_i}).$$

Let $\theta_1, \ldots, \theta_l \in \mathrm{Eq}B$. Then $\Phi(\theta_1, \ldots, \theta_l) = \bigcup_{f \in F_n} \Phi_f(\theta_1, \ldots, \theta_l)$. □

Definition 4.8. Suppose that $\Phi(r_1, \ldots, r_l)$ is the primitive positive formula

$$\langle u_1, u_2\rangle \in \Phi(r_1, \ldots, r_l) \leftrightarrow \exists u_3, \ldots, u_n \bigwedge_{i=1}^{l} r_i(u_{g_i}, u_{h_i}).$$

Define the following term in the language of lattices with a greatest element

$$Q_\Phi(r_1, \ldots, r_l) = \sum_{f \in F_n} T_{\Phi_f}(r_1, \ldots, r_l).$$

Corollary 4.9. *Suppose that $\Phi(r_1, \ldots, r_l)$ is the primitive positive formula*

$$\langle u_1, u_2\rangle \in \Phi(r_1, \ldots, r_l) \leftrightarrow \exists u_3, \ldots, u_n \bigwedge_{i=1}^{l} r_i(u_{g_i}, u_{h_i}).$$

Let $\theta_1, \ldots, \theta_l \in \mathrm{Eq}B$. If $\Phi(\theta_1, \ldots, \theta_l)$ is an equivalence relation on B, then

$$\Phi(\theta_1, \ldots, \theta_l) = Q_\Phi(\theta_1, \ldots, \theta_l).$$

Proof. Suppose that $\Phi(\theta_1,\dots,\theta_l)$ is an equivalence relation. By Corollary 4.7 and Lemma 4.4

$$\Phi(\theta_1,\dots,\theta_l)=\bigcup_{f\in F_n}\Phi_f(\theta_1,\dots,\theta_l)\subseteq\sum_{f\in F_n}T_{\Phi_f}(\theta_1,\dots,\theta_l)=Q_\Phi(\theta_1,\dots,\theta_l).$$

Thus $\Phi(\theta_1,\dots,\theta_l)\subseteq Q_\Phi(\theta_1,\dots,\theta_l)$. Now note that $\Phi_f(\theta_1,\dots,\theta_l)\subseteq\Phi(\theta_1,\dots,\theta_l)$ for each $f\in F_n$ by Corollary 4.7. Therefore, since $\Phi(\theta_1,\dots,\theta_l)$ is an equivalence relation, Lemma 4.4 tells us that $T_{\Phi_f}(\theta_1,\dots,\theta_l)\subseteq\Phi(\theta_1,\dots,\theta_l)$. This now gives that

$$Q_\Phi(\theta_1,\dots,\theta_l)=\sum_{f\in F_n}T_{\Phi_f}(\theta_1,\dots,\theta_l)\subseteq\Phi(\theta_1,\dots,\theta_l).$$

We then have $Q_\Phi(\theta_1,\dots,\theta_l)=\Phi(\theta_1,\dots,\theta_l)$ as desired. □

Definition 4.10. Let $\mathcal{M}=\mathrm{Eq}B$. This lattice is isomorphic to $\mathbf{M}_3$. Let m be a positive integer. Let $\theta_1,\dots,\theta_m$ be equivalence relations on B. By $\langle\theta_1,\dots,\theta_m\rangle$ we will mean the equivalence relation on B^m defined so that $\langle x_1,\dots,x_m\rangle$ is related to $\langle y_1,\dots,y_m\rangle$ precisely when $x_i\theta_i y_i$ for all $i=1,\dots,m$. We will denote the lattice of all equivalence relations on B^m of the form $\langle\theta_1,\dots,\theta_m\rangle$ where each $\theta_i\in\mathcal{M}$ as $\mathcal{M}^m$. Note that this lattice is isomorphic to $\mathbf{M}_3^m$.

Lemma 4.11. *For any positive integer m, every 0–1 sublattice of $\mathcal{M}^m$ is the congruence lattice of an algebra on B^m.*

Proof. Let $\mathcal{L}$ be a 0–1 sublattice of $\mathcal{M}^m$. We will show that $\mathcal{L}$ is closed under primitive positive definitions. Let $\Phi(r_1,\dots,r_l)$ be the primitive positive formula

$$\langle u_1,u_2\rangle\in\Phi(r_1,\dots,r_l)\leftrightarrow\exists u_3,\dots,u_n\bigwedge_{i=1}^{l}r_i(u_{g_i},u_{h_i}).$$

We know from Corollary 4.9 that there is term $Q_\Phi(r_1,\dots,r_l)$ in the language of lattices with a greatest element so that if $\theta_1,\dots,\theta_l\in\mathrm{Eq}B$ and if $\Phi(\theta_1,\dots,\theta_l)$ is an equivalence relation, then $\Phi(\theta_1,\dots,\theta_l)=Q_\Phi^{\mathcal{M}}(\theta_1,\dots,\theta_l)$. Since Q_Φ is a term in the language of lattices with a greatest element, there is a pure lattice term $R_\Phi(r_1,\dots,r_l,y)$ so that $\mathcal{M}$ satisfies $Q_\phi^{\mathcal{M}}(r_1,\dots,r_l)=R_\Phi^{\mathcal{M}}(r_1,\dots,r_l,1)$.

Suppose that $\theta_1,\dots,\theta_l\in\mathcal{L}$ and that $\Phi^{B^m}(\theta_1,\dots,\theta_l)$ is an equivalence relation. For each i, there are equivalence relations $\theta_i^1,\dots,\theta_i^m$ so that $\theta_i=\langle\theta_i^1,\dots,\theta_i^m\rangle$. By virtue of Φ being a primitive positive formula

$$\Phi^{B^m}(\theta_1,\dots,\theta_l)=\langle\Phi^B(\theta_1^1,\dots,\theta_l^1),\dots,\Phi^B(\theta_1^m,\dots,\theta_l^m)\rangle.$$

Since $\Phi^{B^m}(\theta_1, \ldots, \theta_l)$ is an equivalence relation, each $\Phi^B(\theta_1^i, \ldots, \theta_l^i)$ is an equivalence relation on B. Therefore,

$$\begin{aligned}\Phi^{B^m}(\theta_1, \ldots, \theta_l) &= \left\langle \Phi^B(\theta_1^1, \ldots, \theta_l^1), \ldots, \Phi^B(\theta_1^m, \ldots, \theta_l^m)\right\rangle \\ &= \left\langle Q_\Phi^{\mathcal{M}}(\theta_1^1, \ldots, \theta_l^1), \ldots, Q_\Phi^{\mathcal{M}}(\theta_1^m, \ldots, \theta_l^m)\right\rangle \\ &= \left\langle R_\Phi^{\mathcal{M}}(\theta_1^1, \ldots, \theta_l^1, 1), \ldots, R_\Phi^{\mathcal{M}}(\theta_1^m, \ldots, \theta_l^m, 1)\right\rangle \\ &= R_\Phi^{\mathcal{M}^m}(\theta_1, \ldots, \theta_l, \left\langle 1, \ldots, 1\right\rangle).\end{aligned}$$

Since $\mathcal{L}$ is a 0–1 sublattice of $\mathcal{M}^m$, and since R_Φ is a lattice term, this is an element of $\mathcal{L}$. The lattice $\mathcal{L}$ is thus closed under primitive positive definitions which yield equivalence relations, so by 3.1 it is the congruence lattice of an algebra on B^m. $\square$

Every finite lattice in the variety generated by $\mathbf{M}_3$ is isomorphic to a 0–1 sublattice of a finite direct power of $\mathbf{M}_3$. Therefore, Theorem 4.1 now follows directly from Corollary 4.11.

References

[1] V. B. Bodnarchuk, L. A. Kalužnin, V. N. Kotov, and B. A. Romov, *Galois theory for Post algebras i, ii*, Kibernetika (Kiev). **3** (1969), 1–10; **5** (1969), 1–9.

[2] R. Pöschel and L. A. Kalužnin, *Funktionen und Relationenalgebren*, Birkhäuser, Basel, 1979.

[3] P. Pudlák and J. Tůma, *Yeast graphs and fermentation of algebraic lattices*, Coll. Math. Soc. János Bolyai, **14** (1976), 301–341, (Lattice Theory, Szeged, 1974; eds A. P. Huhn and E. T. Schmidt), North Holland.

[4] R. W. Quackenbush and B. Wolk, *Strong representations of congruence lattices*, Algebra Universalis, **1** (1971), 165–166.

[5] J. W. Snow, *A constructive approach to the finite congruence lattice representation problem*, Algebra Universalis, **43** (2000), 279–293.

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