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Primitive positive clones of groupoids

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Abstract. We prove that if two finite groupoids with unity have the same ternary compatible relations, then they have the same primitive positive clones.

1. Introduction

In [3], K. Kearnes and Á. Szendrei prove that if a finite group G has Abelian Sylow subgroups, then the term operations of G are precisely those operations on G which preserve the subgroups of G^3. In contrast to this, they give an example of a ternary operation on the eight-element quaternion group which preserves all compatible ternary relations but is not a term operation. Thus ternary relations may not completely determine the clone of a finite group among all clones on the universe. However, Kearnes and Szendrei ask if these relations may be enough to distinguish between group clones. In [3], they pose the following problem:

Problem 1.1 (Problem 3.18 of [3])**.** Suppose that G and H are groups defined on the same set. Show that $\mathrm{Sub}(G^3) = \mathrm{Sub}(H^3)$ implies $\mathrm{Clo}(G) = \mathrm{Clo}(H)$.

In this paper, we prove that if $\mathbf{A}$ is a finite groupoid with an identity element, then all of the homomorphisms between finite direct products of subalgebras of $\mathbf{A}$ are completely determined by $\mathrm{Sub}(\mathbf{A}^3)$. This implies that $\mathrm{Sub}(\mathbf{A}^3)$ determines the centralizer clone of $\mathbf{A}$ and, hence, the primitive positive clone generated by the operations of $\mathbf{A}$. It follows that the groups G and H in the Kearnes–Szendrei problem have the same primitive positive clones.

2. Preliminaries

A *primitive positive formula* is a first order formula of the form $\exists \wedge$ (atomic). A clone $\mathcal{C}$ on a set A is a *primitive positive clone* if every operation definable on A from operations in $\mathcal{C}$ using primitive positive definitions is already in $\mathcal{C}$. The *primitive positive closure* of a set $\mathcal{F}$ of operations on a set A is the smallest primitive positive

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clone on A containing $\mathcal{F}$. We will denote the primitive positive closure of a set $\mathcal{F}$ of operations as $\mathrm{PPClo}(\mathcal{F})$. If $\mathbf{A} = \langle A, \mathcal{F}\rangle$ is an algebra, then $\mathrm{PPClo}(\mathbf{A})$ will be used to denote $\mathrm{PPClo}(\mathcal{F})$.

For two similar algebras $\mathbf{A}$ and $\mathbf{B}$, we use $\mathrm{Hom}(\mathbf{A}, \mathbf{B})$ to denote the set of homomorphisms from $\mathbf{A}$ to $\mathbf{B}$. For a class $\mathcal{K}$ of similar algebras, we will write $\mathrm{Hom}(\mathcal{K})$ for the collection of all homomorphisms between algebras in $\mathcal{K}$. If $\mathbf{A}$ is similar to the algebras in $\mathcal{K}$, then $\mathrm{Hom}(\mathbf{A}, \mathcal{K})$ is the class of all homomorphisms from $\mathbf{A}$ to an algebra in $\mathcal{K}$.

The *centralizer clone* of an algebra $\mathbf{A}$ is the clone $\mathcal{Z}(\mathbf{A}) = \bigcup_{n\geq 1} \mathrm{Hom}(\mathbf{A}^n, \mathbf{A})$. If $\mathbf{A} = \langle A, \mathcal{F}\rangle$ is an algebra, then we may write $\mathcal{Z}(\mathcal{F})$ for $\mathcal{Z}(\mathbf{A})$. Let $\mathcal{C}$ be a clone on a finite set A. It follows from the Galois connection between operations and relations on a set developed by Bodnarchuk, Kalužnin, Kotov, and Romov in [1] that $\mathrm{PPClo}(\mathcal{C}) = \mathcal{Z}(\mathcal{Z}(\mathcal{C}))$ and that $\mathcal{C}$ is a primitive positive clone if and only if it is a centralizer clone. The next lemma will be our tool for telling when two sets of operations generate the same primitive positive clone. For proofs of these and other fundamental results about primitive positive clones, see [5, 4, 6].

Lemma 2.1 (A. V. Kuznecov [4], see also L. Szabó [5])**.** *Suppose that $\mathcal{F}_1$ and $\mathcal{F}_2$ are sets of operations on a finite set A. Then $\mathrm{PPClo}(\mathcal{F}_1) = \mathrm{PPClo}(\mathcal{F}_2)$ if and only if $\mathcal{Z}(\mathcal{F}_1) = \mathcal{Z}(\mathcal{F}_2)$.*

For any class $\mathcal{K}$ of similar algebras, $\mathrm{P}_n(\mathcal{K})$ will represent the class of all algebras which are direct products of at most n (not necessarily distinct) algebras from $\mathcal{K}$, and $\mathrm{P}_{\mathrm{fin}}(\mathcal{K}) = \bigcup_{n\geq 1} \mathrm{P}_n(\mathcal{K})$.

3. Groupoids

This section is concerned with algebras $\mathbf{A} = \langle A, \cdot, 1\rangle$ with a binary operation and a constant which is an identity element for the binary operation. We will call these algebras *groupoids with identity*. We prove that the centralizer clone of a groupoid $\mathbf{A}$ with identity 1 is completely determined by $\mathrm{Sub}(\mathbf{A}^3)$.

Suppose that $\mathbf{A}$ is an algebra, that $\mathbf{K}_0, \mathbf{K}_1, \dots, \mathbf{K}_n \in \mathrm{Sub}(\mathbf{A})$, and that $\alpha_i \in \mathrm{Con}(\mathbf{K}_i)$ for all i. For any function $F\colon \prod_{i=1}^n K_i/\alpha_i \to K_0/\alpha_0$, define the extended graph of F to be

$$\mathrm{EG}(F) = \big\{\langle x_1, \dots, x_n, y\rangle \in \big(\textstyle\prod_{i=1}^n K_i\big) \times K_0 : F(x_1/\alpha_1, \dots, x_n/\alpha_n) = y/\alpha_0\big\}.$$

Then this lemma is easy to prove:

Lemma 3.1. *Suppose that $\mathbf{A}$ is an algebra, that $\mathbf{K}_0, \mathbf{K}_1, \dots, \mathbf{K}_n \in \mathrm{Sub}(\mathbf{A})$, that $\alpha_i \in \mathrm{Con}(\mathbf{K}_i)$ for all i, and that $F\colon \prod_{i=1}^n K_i/\alpha_i \to K_0/\alpha_0$ is any function. Then F is a homomorphism from $\prod_{i=1}^n \mathbf{K}_i/\alpha_i$ to $\mathbf{K}_0/\alpha_0$ if and only if $\mathrm{EG}(F) \in \mathrm{Sub}(\mathbf{A}^{n+1})$.*

If α is an equivalence relation on a set A, then we will write π_α for the canonical projection from A to A/α.

Theorem 3.2. *Suppose that* $\mathbf{A} = \langle A, \cdot, 1\rangle$ *is a finite groupoid with an identity element. Let* $\overline{\mathbf{A}}$ *be the algebra on the set* A *whose basic operations are all operations which preserve* $\mathrm{Sub}(\mathbf{A}^3)$. *Then the following are true:*

(1) $\mathrm{Hom}(\mathrm{P_{fin}HS}(\mathbf{A})) = \mathrm{Hom}(\mathrm{P_{fin}HS}(\overline{\mathbf{A}}))$.
(2) $\mathcal{Z}(\mathbf{A}) = \mathcal{Z}(\overline{\mathbf{A}})$.
(3) $\mathrm{PPClo}(\mathbf{A}) = \mathrm{PPClo}(\overline{\mathbf{A}})$.

Proof. We prove (1). Then (2) will follow from the definition of the centralizer clone, and (3) will follow from Lemma 2.1. First, note that from the definition of $\overline{\mathbf{A}}$ it follows that $\mathrm{Sub}(\mathbf{A}^3) = \mathrm{Sub}(\overline{\mathbf{A}}^3)$. From this, we know that $\mathrm{Sub}(\mathbf{A}^2) = \mathrm{Sub}(\overline{\mathbf{A}}^2)$ and $\mathrm{Sub}(\mathbf{A}) = \mathrm{Sub}(\overline{\mathbf{A}})$. It follows that $\mathbf{A}$ and $\overline{\mathbf{A}}$ have the same subalgebras and congruences. Thus the universes which appear in $\mathrm{P_{fin}HS}(\mathbf{A})$ and $\mathrm{P_{fin}HS}(\overline{\mathbf{A}})$ are the same (so it even makes sense to ask if (1) is true). For any $\mathbf{B} \in \mathrm{P_{fin}HS}(\mathbf{A})$, we will use $\overline{\mathbf{B}}$ to represent the algebra in $\mathrm{P_{fin}HS}(\overline{\mathbf{A}})$ with the same universe as $\mathbf{B}$.

We will prove by induction on n that $\mathrm{Hom}(\mathbf{B}, \mathrm{HS}(\mathbf{A})) \subseteq \mathrm{Hom}(\overline{\mathbf{B}}, \mathrm{HS}(\overline{\mathbf{A}}))$ for all positive integers n and for every $\mathbf{B} \in \mathrm{P}_n\mathrm{HS}(\mathbf{A})$. This is true for $n = 1, 2$ because $\mathrm{Sub}(\mathbf{A}^3) = \mathrm{Sub}(\overline{\mathbf{A}}^3)$ and since the extended graph of a homomorphism from an algebra in $\mathrm{P_1HS}(\mathbf{A})$ or $\mathrm{P_2HS}(\mathbf{A})$ to an algebra in $\mathrm{HS}(\mathbf{A})$ can be realized in $\mathrm{Sub}(\mathbf{A}^2) = \mathrm{Sub}(\overline{\mathbf{A}}^2)$ or $\mathrm{Sub}(\mathbf{A}^3) = \mathrm{Sub}(\overline{\mathbf{A}}^3)$.

Suppose then that $n \geq 2$ and that $\mathrm{Hom}(\mathbf{B}, \mathrm{HS}(\mathbf{A})) \subseteq \mathrm{Hom}(\overline{\mathbf{B}}, \mathrm{HS}(\overline{\mathbf{A}}))$ for all $\mathbf{B} \in \mathrm{P}_m\mathrm{HS}(\mathbf{A})$ for all $m \leq n$. Let $\mathbf{B} \in \mathrm{P}_{n+1}\mathrm{HS}(\mathbf{A})$ and $\mathbf{C} \in \mathrm{HS}(\mathbf{A})$. There are $\mathbf{K}_0, \mathbf{K}_1, \ldots, \mathbf{K}_{n+1} \in \mathrm{Sub}(\mathbf{A})$ and $\alpha_i \in \mathrm{Con}(\mathbf{K}_i)$ so that $\mathbf{B} = \prod_{i=1}^{n+1} \mathbf{K}_i/\alpha_i$ and $\mathbf{C} = \mathbf{K}_0/\alpha_0$. Let $F\colon \mathbf{B} \to \mathbf{C}$ be a homomorphism. Define

$$F_1\colon \prod_{i=1}^{n} \mathbf{K}_i/\alpha_i \to \mathbf{K}_0/\alpha_0 \quad \text{by} \quad F_1(x_1, \ldots, x_n) = F(x_1, \ldots, x_n, 1)$$

and

$$F_2\colon \mathbf{K}_{n+1}/\alpha_{n+1} \to \mathbf{K}_0/\alpha_0 \quad \text{by} \quad F_2(x_{n+1}) = F(1, \ldots, 1, x_{n+1}).$$

Then F_1 and F_2 are homomorphisms. For $i = 1, 2$, let $\mathbf{H}_i \leq \mathbf{K}_0$ be given by $\mathbf{H}_i = \pi_{\alpha_0}^{-1}(\mathrm{Im}(F_i))$. And define

$$G\colon \mathbf{H}_1/\alpha_0 \times \mathbf{H}_2/\alpha_0 \to \mathbf{K}_0/\alpha_0 \quad \text{by} \quad G(x, y) = x \cdot y.$$

Then G is also a homomorphism. To see this, let $(a, b), (c, d) \in \mathbf{H}_1/\alpha_0 \times \mathbf{H}_2/\alpha_0$. There are $\langle x_1, \ldots, x_n\rangle, \langle u_1, \ldots, u_n\rangle \in \prod_{i=1}^{n} \mathbf{K}_i/\alpha_i$ so that $F_1(x_1, \ldots, x_n) = a$ and $F_1(u_1, \ldots, u_n) = c$. Also, there are $x_{n+1}, u_{n+1} \in \mathbf{K}_{n+1}/\alpha_{n+1}$ so that $F_2(x_{n+1}) = b$

and $F_2(u_{n+1}) = d$. Then

$$\begin{aligned}
G(a,b)G(c,d) &= (a \cdot b) \cdot (c \cdot d) \\
&= (F_1(x_1,\ldots,x_n) \cdot F_2(x_{n+1})) \cdot (F_1(u_1,\ldots,u_n) \cdot F_2(u_{n+1})) \\
&= (F(x_1,\ldots,x_n,1) \cdot F(1,\ldots,1,x_{n+1})) \cdot (F(u_1,\ldots,u_n,1) \cdot F(1,\ldots,1,u_{n+1})) \\
&= F(x_1 \cdot u_1,\ldots,x_n \cdot u_n, x_{n+1} \cdot u_{n+1}) \\
&= F(x_1 \cdot u_1,\ldots,x_n \cdot u_n, 1) \cdot F(1,\ldots,1,x_{n+1} \cdot u_{n+1}) \\
&= (F(x_1,\ldots,x_n,1) \cdot F(u_1,\ldots,u_n,1)) \cdot (F(1,\ldots,1,x_{n+1}) \cdot F(1,\ldots,1,u_{n+1})) \\
&= (a \cdot c) \cdot (b \cdot d) = G(a \cdot c, b \cdot d).
\end{aligned}$$

Thus G is also a homomorphism. Since F_1, F_2, and G are homomorphisms for the algebraic structures induced by $\mathbf{A}$, we can assume by induction that F_1, F_2, and G are homomorphisms for the algebraic structures induced by $\overline{\mathbf{A}}$. This means by Lemma 3.1 that the extended graphs $\mathrm{EG}(F_1)$, $\mathrm{EG}(F_2)$, and $\mathrm{EG}(G)$ are closed under the operations induced by $\overline{\mathbf{A}}$. If a collection of relations on an algebra is closed under the operations of the algebra, then so is every relation definable from those relations by a primitive positive formula [1]. Then the following relation R defined by a primitive positive formula from these extended graphs is also compatible with $\overline{\mathbf{A}}$:

$$\begin{aligned}
&\langle x_1,\ldots,x_n,x_{n+1},y\rangle \in R \leftrightarrow (\exists a,b \in A) \\
&\qquad \big([\langle x_1,\ldots,x_n,a\rangle \in \mathrm{EG}(F_1)] \wedge [\langle x_{n+1},b\rangle \in \mathrm{EG}(F_2)] \wedge [(a,b,y) \in \mathrm{EG}(G)]\big).
\end{aligned}$$

It is easy to see that

$$\langle x_1,\ldots,x_n,x_{n+1},y\rangle \in R$$

if and only if

$$F_1(x_1/\alpha_1,\ldots,x_n/\alpha_n) \cdot F_2(x_{n+1}/\alpha_{n+1}) = y/\alpha_0.$$

However,

$$F_1(x_1/\alpha_1,\ldots,x_n/\alpha_n) \cdot F_2(x_{n+1}/\alpha_{n+1}) = F(x_1/\alpha_1,\ldots,x_n/\alpha_n,x_{n+1}/\alpha_{n+1}),$$

so R is nothing other than $\mathrm{EG}(F)$. Since $\mathrm{EG}(F) = R$ is compatible with $\overline{\mathbf{A}}$, it follows that F is a homomorphism from $\prod_{i=1}^n \overline{\mathbf{K}}_i/\alpha_i$ to $\overline{\mathbf{K}}_0/\alpha_0$.

By induction we now have that, for each $n = 1,2,\ldots$ and for each $\mathbf{B} \in \mathrm{P}_n\mathrm{HS}(\mathbf{A})$, $\mathrm{Hom}(\mathbf{B},\mathrm{HS}(\mathbf{A})) \subseteq \mathrm{Hom}(\overline{\mathbf{B}},\mathrm{HS}(\mathbf{A}))$. It follows that $\mathrm{Hom}(\mathrm{P_{fin}HS}(\mathbf{A}),\mathrm{HS}(\mathbf{A})) \subseteq \mathrm{Hom}(\mathrm{P_{fin}HS}(\overline{\mathbf{A}}),\mathrm{HS}(\overline{\mathbf{A}}))$. Now, since every function into a product is uniquely determined by its component maps, and since such a function is a homomorphism if and only if its component maps are, we have that $\mathrm{Hom}(\mathrm{P_{fin}HS}(\mathbf{A})) \subseteq \mathrm{Hom}(\mathrm{P_{fin}HS}(\overline{\mathbf{A}}))$. Of course, since the operations of $\mathbf{A}$ are also operations of $\overline{\mathbf{A}}$, the reverse inclusion also holds, so we have (1). The statements (2) and (3) now follow as mentioned. □

Note that finiteness only plays a role in the proof of Theorem 3.2 in the reference to Lemma 2.1 to establish (3). Thus (1) and (2) also hold when **A** is infinite.

Suppose now that **A** and **B** are finite groupoids with identity on the same universe and that $\mathrm{Sub}(\mathbf{A}^3) = \mathrm{Sub}(\mathbf{B}^3)$. Let $\overline{\mathbf{A}}$ be the algebra with the same universe as **A** and **B** whose operations are all operations preserving the relations in $\mathrm{Sub}(\mathbf{A}^3) = \mathrm{Sub}(\mathbf{B}^3)$. Then by the previous theorem

$$\mathrm{Hom}(\mathrm{P_{fin}HS}(\mathbf{A})) = \mathrm{Hom}(\mathrm{P_{fin}HS}(\overline{\mathbf{A}})) = \mathrm{Hom}(\mathrm{P_{fin}HS}(\mathbf{B})),$$
$$\mathcal{Z}(\mathbf{A}) = \mathcal{Z}(\overline{\mathbf{A}}) = \mathcal{Z}(\mathbf{B}),$$
$$\mathrm{PPClo}(\mathbf{A}) = \mathrm{PPClo}(\overline{\mathbf{A}}) = \mathrm{PPClo}(\mathbf{B}).$$

We have:

Theorem 3.3. *Suppose that* **A** *and* **B** *are finite groupoids with identity on the same universe. If* $\mathrm{Sub}(\mathbf{A}^3) = \mathrm{Sub}(\mathbf{B}^3)$*, then*

(1) $\mathrm{Hom}(\mathrm{P_{fin}HS}(\mathbf{A})) = \mathrm{Hom}(\mathrm{P_{fin}HS}(\mathbf{B}))$.
(2) $\mathcal{Z}(\mathbf{A}) = \mathcal{Z}(\mathbf{B})$.
(3) $\mathrm{PPClo}(\mathbf{A}) = \mathrm{PPClo}(\mathbf{B})$.

We can say a bit more in the presence of associativity and a little commutativity.

Corollary 3.4. *Suppose that* $\mathbf{A}_1 = \langle A, \cdot, 1\rangle$ *and* $\mathbf{A}_2 = \langle A, *, 1\rangle$ *are finite groupoids with identity defined on the same universe and that* $\mathrm{Sub}(\mathbf{A}_1^3) = \mathrm{Sub}(\mathbf{A}_2^3)$*. Suppose also that* $\mathbf{A}_1$ *is associative. Then*

(1) *If* x *and* y *commute in* $\mathbf{A}_1$*, then* $x * y = y * x = x \cdot y$*.*
(2) *Positive powers agree in* $\mathbf{A}_1$ *and* $\mathbf{A}_2$*.*
(3) *If* x *and* y *are inverses in* $\mathbf{A}_1$*, then* x *and* y *are inverses in* $\mathbf{A}_2$*.*

Proof. Let **H** and **K** be the subgroupoids (with identity) of $\mathbf{A}_1$ generated by x and y, respectively. Note that by using associativity and the commutativity of x and y, every element of **H** commutes with every element of **K** (under the operation of $\mathbf{A}_1$). Define a function $f\colon H \times K \to A$ by $f(a,b) = a \cdot b$. Then f is an $\mathbf{A}_1$-homomorphism. To see this, let $\langle h_1, k_1\rangle, \langle h_2, k_2\rangle \in H \times K$. Then

$$\begin{aligned} f(h_1,k_1)\cdot f(h_2,k_2) &= (h_1\cdot k_1)\cdot(h_2\cdot k_2) = h_1\cdot((k_1\cdot h_2)\cdot k_2) \\ &= h_1\cdot((h_2\cdot k_1)\cdot k_2) = (h_1\cdot h_2)\cdot(k_1\cdot k_2) \\ &= f(h_1\cdot h_2, k_1\cdot k_2) = f(\langle h_1,k_1\rangle\cdot\langle h_2,k_2\rangle). \end{aligned}$$

Notice that what is essential here is the commutativity and associativity of elements of **H** and **K**. Since f is an $\mathbf{A}_1$-homomorphism, it follows from Theorem 3.3 that f

is an $\mathbf{A}_2$-homomorphism. Then

$$\begin{aligned} x * y &= f(x,1) * f(1,y) = f(\langle x,1\rangle * \langle 1,y\rangle) = f(x,y) \\ &= f(\langle 1,y\rangle * \langle x,1\rangle) = f(1,y) * f(x,1) = y * x \end{aligned}$$

and

$$x * y = f(x,y) = x \cdot y.$$

This proves (1). Now (2) follows from (1) since any element must commute with its positive powers in $\mathbf{A}_1$. Statement (3) also follows from (1) since any element with an inverse must commute with its inverse. □

We close by noting that our results do not solve even the finite version of the Kearnes–Szendrei problem mentioned in the introduction because the primitive positive clone of a finite group may or may not be equal to the clone of the group. If G is a finite simple non-Abelian group, then the primitive positive clone of G contains the ternary discriminator operation by Lemma 2.2 of [2]. In this case, the primitive positive clone of G is larger than $\mathrm{Clo}(G)$. On the other hand, if G is a *finitely generated* Abelian group, then $\mathcal{Z}(\mathcal{Z}(G)) = \mathrm{Clo}(G)$, so if G is finite, $\mathrm{PPClo}(G) = \mathrm{Clo}(G)$. To see this, suppose that G is a finitely generated Abelian group. We know $\mathrm{Clo}(G) \subseteq \mathcal{Z}(\mathcal{Z}(G))$, so we just need to show the opposite inclusion. Since the binary operation of G is in $\mathcal{Z}(G)$, every member of $\mathcal{Z}(\mathcal{Z}(G))$ is a homomorphism from a direct power of G into G. Every such homomorphism is a sum of coordinate homomorphisms which must also be in $\mathcal{Z}(\mathcal{Z}(G))$. Thus, to show $\mathcal{Z}(\mathcal{Z}(G)) \subseteq \mathrm{Clo}(G)$, it is enough to show that every unary member of $\mathcal{Z}(\mathcal{Z}(G))$ is in $\mathrm{Clo}(G)$. By the Fundamental Theorem of Finitely Generated Abelian Groups, we can assume that G is a direct product

$$\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k} \times \mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$$

with n_i a factor of n_{i+1} for $i = 1, 2, \ldots, k-1$. Suppose that there are m factors here. For each $i = 1, 2, \ldots, m$, let e_i be the vector in G with 1 in the i^{th} coordinate and 0 everywhere else. For each $i \leq m$, there is an endomorphism g_i of G so that $g_i(e_m) = e_i$ and $g_i(e_j) = 0$ else. Notice that if $x \in G$, then $g_i(x) = le_i$ for some integer l. Let f be a unary function in $\mathcal{Z}(\mathcal{Z}(G))$. Since f and g_i commute, $f(e_i) = f(g_i(e_m)) = g_i(f(e_m))$. Thus, there is some integer l_i with $f(e_i) = l_i e_i$. Now notice that

$$l_m e_i = l_m g_i(e_m) = g_i(l_m e_m) = g_i(f(e_m)) = f(g_i(e_m)) = f(e_i)$$

so $f(e_i) = l_m e_i$ for all i. Now let $x = \langle x_1, x_2, \ldots, x_m\rangle \in G$. Then

$$f(x) = f\Big(\sum_{i=1}^{m} x_i e_i\Big) = \sum_{i=1}^{m} x_i f(e_i) = \sum_{i=1}^{m} x_i l_m e_i = l_m \sum_{i=1}^{m} x_i e_i = l_m x.$$

Thus $f \in \mathrm{Clo}(G)$. Since every operation in $\mathcal{Z}(\mathcal{Z}(G))$ is a sum of unary operations of $\mathcal{Z}(\mathcal{Z}(G))$ applied coordinatewise, it follows that $\mathcal{Z}(\mathcal{Z}(G)) \subseteq \mathrm{Clo}(G)$ and that these sets are equal.

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