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Almost distributive sublattices and congruence heredity

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Abstract. A congruence lattice **L** of an algebra **A** is hereditary if every 0-1 sublattice of **L** is the congruence lattice of an algebra on A. Suppose that **L** is a finite lattice obtained from a distributive lattice by doubling a convex subset. We prove that every congruence lattice of a finite algebra isomorphic to **L** is hereditary.

1. Introduction

If **L** is the congruence lattice of a finite algebra **A** and every 0-1 sublattice of **L** is also the congruence lattice of an algebra with the same universe as **A**, then **L** is called a hereditary congruence lattice. Furthermore, if every 0-1 sublattice of $\mathbf{L}^n$ is the congruence lattice of an algebra on A^n for every positive integer n, then **L** is a power-hereditary congruence lattice. These concepts were introduced in [5] by Hegedűs and Pálfy. In that manuscript a complete characterization is given of all Abelian prime power order groups whose congruence lattices are (power-)hereditary.

In [8] the author proves that every congruence lattice representation of $\mathbf{N}_5$ is power-hereditary. The lattice $\mathbf{N}_5$ can be obtained from the four element boolean algebra by doubling one of the atoms. We give a partial extension of the result of [8] in this manuscript by proving that any congruence lattice representation of a finite lattice obtained by doubling a convex subset of a distributive lattice is hereditary. We do not know if "hereditary" here can be replaced by "power-hereditary."

Lemmas 4.4 and 4.5 of [5] can be interpreted as saying that whether or not a congruence lattice is (power-)hereditary is a matter of whether or not certain first order defined operations on the congruence lattice can be interpolated with lattice terms and whether this interpolation is local or global. We prove that if **L** is the congruence lattice of a finite algebra and $\delta \in \mathrm{Con}\mathbf{L}$ with $\mathbf{L}/\delta$ distributive, then every one of these operations can be interpolated by a lattice term globally modulo δ. As a result, every sublattice of **L** which is constructed in a regular way from δ classes is a congruence lattice. These are the "almost distributive" sublattices

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referenced in the title. This is a generalization of Theorem 3.2 of [8] which says that every subdirect product of a distributive lattice and a congruence lattice is again a congruence lattice.

2. Preliminaries

If α is a binary relation on a set A and β is a binary relation on a set B, then the relation $\langle \alpha, \beta \rangle$ is a binary relation on $A \times B$ defined so that $\langle a_1, b_1 \rangle \langle \alpha, \beta \rangle \langle a_2, b_2 \rangle$ if and only if $a_1 \alpha a_2$ and $b_1 \beta b_2$. If $\mathbf{L}$ is a lattice of equivalence relations on a set A and $\mathbf{M}$ is a lattice of equivalence relations on a set B, then $\mathbf{L} \times \mathbf{M}$ is the lattice of all equivalence relations on $A \times B$ of the form $\langle \alpha, \beta \rangle$ where $\alpha \in \mathbf{L}$ and $\beta \in \mathbf{M}$. These definitions extend naturally to direct powers $\mathbf{L}^n$ of lattices of equivalence relations. If $\mathbf{L}$ is the congruence lattice of a finite algebra $\mathbf{A}$ and every 0-1 sublattice of $\mathbf{L}$ is also the congruence lattice of an algebra with the same universe as $\mathbf{A}$, then $\mathbf{L}$ is called a *hereditary congruence lattice.* Furthermore, if every 0-1 sublattice of $\mathbf{L}^n$ is the congruence lattice of an algebra on A^n for all positive integers n, then $\mathbf{L}$ is a *power-hereditary congruence lattice.*

By a *representation* or a *congruence representation* of a finite lattice $\mathbf{L}$ we will mean the congruence lattice $\mathrm{Con}\mathbf{A}$ of a finite algebra such that $\mathrm{Con}\mathbf{A} \cong \mathbf{L}$. If $\mathrm{Con}\mathbf{A}$ is a representation of $\mathbf{L}$ and $\mathrm{Con}\mathbf{A}$ is a (power-)hereditary congruence lattice, then we will say that $\mathrm{Con}\mathbf{A}$ is a (power-)hereditary representation.

A *primitive positive formula* is a formula of the form $\exists \wedge$ (atomic). If Φ is a primitive positive formula employing binary relation symbols $r_1, \dots, r_n$ and if Φ has two free variables, then Φ naturally induces an operation on the set of binary relations of any set. If $\theta_1, \dots, \theta_n$ are binary relations on a set A, then we will use $\Phi(\theta_1, \dots, \theta_n)$ to represent the binary relation on A defined by interpreting each r_i in Φ as θ_i. The operation $\langle \theta_1, \dots, \theta_n \rangle \mapsto \Phi(\theta_1, \dots, \theta_n)$ is order preserving, and when it is applied to products of relations can be applied coordinate-wise.

In [9] the author proves that the set of all representable finite lattices is closed under certain lattice theoretic operations. The main tool exploited there is the following lemma, which follows from the fact that a set of relations on a finite set is the set of all relations compatible with an algebra on the set if and only if the relations are closed under primitive positive definitions [1, 6].

Lemma 2.1 ([9, Corollary 2.2])**.** *Suppose* $\mathbf{L}$ *is a* 0-1 *lattice of equivalence relations on a finite set* A. *There is an algebra* $\mathbf{A}$ *on* A *with* $\mathrm{Con}\mathbf{A} = \mathbf{L}$ *if and only if every equivalence relation on* A *which can be defined from* $\mathbf{L}$ *by a primitive positive formula is already in* $\mathbf{L}$.

Suppose also that $\mathbf{G}$ is a finite directed graph with vertices labelled $x_0, \dots, x_m$ and edges labelled $r_1, \dots, r_n$. To indicate that there is an edge from x_i to x_k in $\mathbf{G}$

labelled by r_j, we write $x_i \xrightarrow{r_j} y_k$. Let $\Phi_{\mathbf{G}}$ be the primitive positive formula

$$\exists x_2, \ldots, x_m \bigwedge_{x_i \xrightarrow{r_j} y_k} r_j(x_i, x_k).$$

If $r_1^A, \ldots, r_n^A$ are binary relations on a finite set A, then in the language of [12] the relation $\Phi_{\mathbf{G}}(r_1^A, \ldots, r_n^A)$ is called the *graphical composition* of $r_1, \ldots, r_n$ associated with the labelling of each r_i edge in $\mathbf{G}$ by r_i^A. Every graphical composition is a primitive positive definition.

Suppose that Φ is a primitive positive formula using binary relation symbols $r_1, \ldots, r_n$ which has exactly two free variables. Let $x_0, \ldots, x_m$ be the variables in Φ. By the *graph of* $\Phi(r_1, \ldots, r_n)$ we will mean the directed graph $\mathbf{G}_\Phi$ with vertices $\{x_0, \ldots, x_n\}$ so that for each occurrence of $r_i(x_j, x_k)$ in Φ, there is an edge in $\mathbf{G}$ labelled by r_i extending from x_j to x_k. The maps $\Phi \to \mathbf{G}_\Phi$ and $\mathbf{G} \to \Phi_{\mathbf{G}}$ give a natural one-to-one correspondence between primitive positive formulas using binary relations symbols with two free variables and graphical compositions. In [12], it is proven in Theorem 2.6 that a lattice of equivalence relations on a finite set is a congruence lattice if and only if that lattice is closed under graphical compositions which yield equivalence relations. This is equivalent to our Lemma 2.1.

In the environment of binary relations, primitive positive definitions and graphical compositions are essentially one and the same. There will be times in this manuscript when the notions inherent in graphical compositions (paths, connectedness, etc.) will make it easier to visualize our arguments. Most of the time however, the actual structure of the associated graph beyond connectedness will not matter. In these instances, the generic notation of primitive positive formulas is adequate. In a more general setting, primitive positive formulas do have the advantage that they can be extended naturally to working with relations of any rank (as in [11]) and even to working with clones (as in [10]). These extensions will not be needed here, where we are dealing with equivalence relations. We will assume from here on that every primitive positive formula only contains binary relation symbols and has exactly two free variables. A primitive positive formula Φ will be called *connected* if the corresponding graph is connected. The proof of Lemma 3.1 of [9] establishes:

Lemma 2.2. *Suppose that $\alpha < \beta$ are equivalence relations on a finite set A and that $\Phi(r_1, \ldots, r_n)$ is a connected primitive positive formula. If $R_1, \ldots, R_n \in \{\alpha, \beta\}$ then $\Phi(R_1, \ldots, R_n) \in \{\alpha, \beta\}$. Moreover, $\Phi(R_1, \ldots, R_n) = \alpha$ if and only if there is a path between the free variables in the graph of $\Phi(R_1, \ldots, R_n)$ labelled by α.*

Suppose that $r_1, \ldots, r_n$ are congruences on a finite algebra $\mathbf{A}$ and that Φ is a primitive positive formula so that $r = \Phi(r_1, \ldots, r_n)$ is an equivalence relation. If Con$\mathbf{A}$ is hereditary, then r must be in the sublattice of Con$\mathbf{A}$ generated by

$r_1, \ldots, r_n$. This means that there is a lattice term T so that $r = T(r_1, \ldots, r_n)$. On the other hand, if such a term always exists, then Con$\mathbf{A}$ must be hereditary.

Lemma 2.3. *The congruence lattice of a finite algebra* $\mathbf{A}$ *is hereditary if and only if for every primitive positive formula* $\Phi(r_1, \ldots, r_n)$ *and for all* $R_1, \ldots, R_n \in$ Con$\mathbf{A}$ *if* $\Phi(R_1, \ldots, R_n)$ *is an equivalence relation, there is a lattice term* T *so that* $T(R_1, \ldots, R_n) = \Phi(R_1, \ldots, R_n)$.

Hegedűs and Pálfy argue that a congruence lattice is power-hereditary if and only if primitive positive definitions can be interpolated globally by lattice terms. Their manuscript uses the notion of graphical composition. We translate their result here into the language of primitive positive formulas.

Lemma 2.4 (See [5, Lemma 4.5])**.** *The congruence lattice of a finite algebra* $\mathbf{A}$ *is power-hereditary if and only if for every primitive positive formula* $\Phi(r_1, \ldots, r_n)$ *there is a lattice term* T *so that if* $R_1, \ldots, R_n \in$ Con$\mathbf{A}$ *and* $\Phi(R_1, \ldots, R_n)$ *is an equivalence relation, then* $T(R_1, \ldots, R_n) = \Phi(R_1, \ldots, R_n)$.

Suppose that $\mathbf{A}$ and $\mathbf{B}$ are finite algebras and $f\colon \mathrm{Con}\mathbf{A} \to \mathrm{Con}\mathbf{B}$ is any function. We will say that f *preserves connected primitive positive definitions* if whenever $\Phi(x_1, \ldots, x_n)$ is a connected primitive positive formula and $r_1, \ldots, r_n \in$ Con$\mathbf{A}$ so that $\Phi(r_1, \ldots, r_n)$ and $\Phi(f(r_1), \ldots, f(r_n))$ are equivalence relations, then $f(\Phi(r_1, \ldots, r_n)) = \Phi(f(r_1), \ldots, f(r_n))$.

Suppose that $\Phi(r_1, \ldots, r_k)$ is a primitive positive formula. It will be useful to have a primitive positive formula which will always give an equivalence relation when evaluated with equivalence relations on a given set and which will agree with Φ wherever Φ already yields an equivalence relation on that set. Assume that the free variables in Φ are x_0 and x_1. By $R(x_0, x_1)$ we will mean the statement that the ordered pair $\langle x_0, x_1\rangle$ is in the relation $\Phi(r_1, \ldots, r_k)$. Note that $R(x_0, x_1)$ is (equivalent to) a primitive positive formula which uses the relation symbols $r_1, \ldots, r_k$. By $\overline{\Phi}^n$ we will mean the primitive positive formula with relation symbols $r_1, \ldots, r_k$ defined so that $\langle a, b\rangle \in \overline{\Phi}^n(r_1, \ldots, r_k)$ if and only if

$$\exists y_0, \ldots, y_{n+1} \Big(\bigwedge_{i=0}^{n} R(y_i, y_{i+1}) \wedge R(y_{i+1}, y_i) \Big) \wedge [(y_0 = a) \wedge (y_{n+1} = b)]$$

If $r_1, \ldots, r_k$ are equivalence relations on a set with no more then n elements, then $\overline{\Phi}^n(r_1, \ldots, r_k)$ is the transitive closure of the largest symmetric relation contained in $\Phi(r_1, \ldots, r_k)$. It is easily seen to be reflexive. To see that a particular x is related to itself via this relation, one can take all of the existentially quantified variables to be equal to x (this works since each r_i is reflexive). Thus $\overline{\Phi}^n(r_1, \ldots, r_k)$ is an equivalence relation. Moreover, if $\Phi(r_1, \ldots, r_k)$ is an equivalence relation then the largest symmetric relation contained in $\Phi(r_1, \ldots, r_k)$ is all of $\Phi(r_1, \ldots, r_k)$, so $\overline{\Phi}^n(r_1, \ldots, r_k) = \Phi(r_1, \ldots, r_k)$.

3. Homomorphisms into distributive lattices

We show that any homomorphism from any congruence lattice into a distributive congruence lattice preserves connected primitive positive definitions. We write **2** for the two element lattice. Note that $\mathrm{Con}\mathbf{2} \cong \mathbf{2}$.

Lemma 3.1. *Suppose that* $\mathbf{A}$ *is a finite algebra and* $f\colon \mathrm{Con}\mathbf{A} \to \mathrm{Con}\mathbf{2}$ *is a surjective homomorphism. Then* f *preserves connected primitive positive definitions.*

Proof. There are elements $a, b \in \mathrm{Con}\mathbf{A}$ so that

$$f(x) = \begin{cases} 0 & x \leq a, \\ 1 & x \geq b. \end{cases}$$

Suppose that $\Phi(x_1, \dots, x_n)$ is a connected primitive positive definition and that $r_1, \dots, r_n$ are congruences on $\mathbf{A}$ with $\Phi(r_1, \dots, r_n) \in \mathrm{Con}\mathbf{A}$. We will show that

$$f(\Phi(r_1, \dots, r_n)) = \Phi(f(r_1), \dots, f(r_n)).$$

The forward inclusion is always true. We prove the reverse. Suppose first that $\Phi(f(r_1), \dots, f(r_n)) = 0$. Then by Lemma 2.2 there is a path between the free variables in the graph of $\Phi(f(r_1), \dots, f(r_n))$ labelled by 0. This means that there is a path between the free variables in the graph of $\Phi(r_1, \dots, r_n)$ labelled by relations below a. But then

$$\Phi(r_1, \dots, r_n) \leq a \text{ and } f(\Phi(r_1, \dots, r_n)) = 0 = \Phi(f(r_1), \dots, f(r_n)).$$

Suppose now that $\Phi(f(r_1), \dots, f(r_n)) = 1$. For each i, let $r'_i = b$ if $r_i \geq b$ and let $r'_i = 0$ otherwise. Since $\Phi(f(r_1), \dots, f(r_n)) = 1$, by Lemma 2.2 there is no path between the free variables in the graph of $\Phi(f(r_1), \dots, f(r_n))$ labelled by 0. This means that in the graph of $\Phi(r'_1, \dots, r'_n)$ there is no path between the free variables labelled by 0. By 2.2, $\Phi(r'_1, \dots, r'_n) = b$. But then $\Phi(r_1, \dots, r_n) \geq \Phi(r'_1, \dots, r'_n) = b$ and $f(\Phi(r_1, \dots, r_n)) = 1 = \Phi(f(r_1), \dots, f(r_n))$. □

This extends easily to homomorphisms into any distributive congruence lattice.

Corollary 3.2. *Suppose that* $\mathbf{A}$ *and* $\mathbf{B}$ *are finite algebras with* $\mathrm{Con}\mathbf{B}$ *distributive. Any homomorphism from* $\mathrm{Con}\mathbf{A}$ *to* $\mathrm{Con}\mathbf{B}$ *must preserve connected primitive positive definitions.*

Proof. Suppose that Φ is a connected primitive positive formula and that $r_1, \dots, r_n$ are congruences on $\mathbf{A}$ with $\Phi(r_1, \dots, r_n)$ and $\Phi(f(r_1), \dots, f(r_n))$ equivalence relations. Let $a = f(\Phi(r_1, \dots, r_n))$ and $b = \Phi(f(r_1), \dots, f(r_n))$. If $a \neq b$, then there is a homomorphism $g\colon \mathrm{Con}\mathbf{B} \to \mathrm{Con}\mathbf{2}$ with $g(a) \neq g(b)$. Now, by the previous lemma, $g(b) = g(\Phi(f(r_1), \dots, f(r_n))) = \Phi(gf(r_1), \dots, gf(r_n))$. But then

$gf\colon \mathrm{Con}\mathbf{A} \to \mathrm{Con}\mathbf{2}$ is a surjective homomorphism with

$$gf(\Phi(r_1,\ldots,r_n) = g(a) \neq g(b) = \Phi(gf(r_1),\ldots,gf(r_n))$$

contrary to the previous lemma. □

4. $\delta_{\mathbf{L}}$-convexity

If $\mathbf{L}$ is a lattice and $\{\alpha_i : i \in I\}$ is a family of congruences on $\mathbf{L}$ so that $\mathbf{L}/\alpha_i$ is distributive for all i, then $\mathbf{L}/(\cap\{\alpha_i : i \in I\})$ is also distributive. This allows us to make the following definition.

Definition 4.1. For any lattice $\mathbf{L}$, let $\delta_{\mathbf{L}}$ be the least congruence on $\mathbf{L}$ so that $\mathbf{L}/\delta_{\mathbf{L}}$ is distributive.

The following two lemmas are not hard to prove.

Lemma 4.2. *Suppose that $\mathbf{L}$ is a lattice and $\alpha \in \mathrm{Con}\mathbf{L}$. Then $\mathbf{L}/\alpha$ is distributive if and only if $\delta_{\mathbf{L}} \leq \alpha$.*

Lemma 4.3. *Suppose that $\mathbf{L}$ is a lattice. Then*

$$\delta_{\mathbf{L}} = \bigcap\{\alpha \in \mathrm{Con}\mathbf{L} : |\mathbf{L}/\alpha| = 2\}$$

(where the intersection over the empty set is taken to be the universal relation).

We first prove that all primitive positive formulas can be interpolated by lattice terms modulo $\delta_{\mathbf{L}}$.

Theorem 4.4. *Let $\mathbf{A}$ be a finite algebra and Φ a connected primitive positive formula. There is a lattice term T so that for all $r_1,\ldots,r_n \in \mathrm{Con}\mathbf{A}$, if $\Phi(r_1,\ldots,r_n)$ is an equivalence relation, then*

$$\Phi(r_1,\ldots,r_n)\delta_{\mathrm{Con}\mathbf{A}}T(r_1,\ldots,r_n).$$

Proof. We will write δ for $\delta_{\mathrm{Con}\mathbf{A}}$. Since $(\mathrm{Con}\mathbf{A})/\delta$ is distributive, we can find a finite algebra $\mathbf{B}$ with $\mathrm{Con}\mathbf{B} \cong (\mathrm{Con}\mathbf{A})/\delta$. Since $\mathrm{Con}\mathbf{B}$ is distributive, it is power-hereditary. (Every finite distributive lattice is the congruence lattice of a finite algebra, and every 0-1 distributive lattice of equivalence relations on a finite set is a congruence lattice by [7], so every distributive congruence lattice is power-hereditary). Let m be the larger of $|A|$ and $|B|$. By Lemma 2.4, there is a lattice term T which is equal to $\overline{\Phi}^m$ on $\mathrm{Con}\mathbf{B}$. Let $r_1,\ldots,r_n \in \mathrm{Con}\mathbf{A}$ and suppose that $\Phi(r_1,\ldots,r_n)$ is an equivalence relation. Let $f\colon \mathrm{Con}\mathbf{A} \to \mathrm{Con}\mathbf{B}$ be a surjective homomorphism with $\ker f = \delta$. Note that by Lemma 3.2, f preserves Φ. Then

$$\begin{aligned} f(\Phi(r_1,\dots,r_n)) &= f(\overline{\Phi}^m(r_1,\dots,r_n)) \\ &= \overline{\Phi}^m(f(r_1),\dots,r(r_n)) \\ &= T(f(r_1),\dots,r(r_n)) \\ &= f(T(r_1,\dots,r_n)). \end{aligned}$$

Hence we have $\Phi(r_1,\dots,r_n)\delta T(r_1,\dots,r_n)$. □

The fact that we can interpolate primitive positive formulas with lattice terms modulo δ implies that sublattices of a congruence lattice which are *built from* δ *classes* in a regular way should be closed under primitive positive definition.

Definition 4.5. Suppose that $\mathbf{M}$ is a sublattice of a lattice $\mathbf{L}$ and $\alpha \in \mathrm{Con}\mathbf{L}$. $\mathbf{M}$ is α*-convex* if for all $x, y, z \in \mathbf{L}$ if $x, z \in \mathbf{M}$, $x\alpha z$ and $x < y < z$, then $y \in \mathbf{M}$.

Theorem 4.6. *Suppose* $\mathbf{L}$ *is the congruence lattice of a finite algebra* $\mathbf{A}$*. Every* $\delta_{\mathbf{L}}$*-convex* 0-1 *sublattice of* $\mathbf{L}$ *is the congruence lattice of an algebra with the same universe as* $\mathbf{A}$*.*

Proof. We will prove this statement by induction on n:

> $\Phi(n)$: Suppose that $\mathbf{N}$ is an n element lattice of equivalence relations on A and that $\mathbf{N}$ is closed under connected primitive positive definitions which yield equivalence relations. Then every $\delta_{\mathbf{N}}$ convex sublattice of $\mathbf{N}$ is also closed under such primitive positive definitions.

This will be enough to establish the theorem. First, $\Phi(1)$ and $\Phi(2)$ are trivially true. Assume that $|\mathbf{L}| = n$ and that $\Phi(m)$ is true for all $2 \le m < n$. Let $f_1, \dots, f_k$ be all surjective homomorphisms from $\mathbf{L}$ to $\mathrm{Con}\mathbf{2}$. Define $F\colon \mathbf{L} \to (\mathrm{Con}\mathbf{2})^k$ by $F(x) = \langle f_1(x), \dots, f_k(x)\rangle$. From 3.2 and 4.3, we know that F preserves primitive positive definitions yielding equivalence relations and that $\ker F = \delta_{\mathbf{L}}$. From here on, write δ for $\delta_{\mathbf{L}}$.

Now suppose that $\mathbf{M}$ is a δ-convex sublattice of $\mathbf{L}$. Let Φ be a connected primitive positive formula, and let $r_1, \dots, r_n \in \mathbf{M}$ so that $r = \Phi(r_1, \dots, r_n)$ is an equivalence relation. Since $\mathbf{L}$ is a congruence lattice and r is defined from relations in $\mathbf{L}$ using a primitive postive formula, we know that $r \in \mathbf{L}$. We must prove that $r \in \mathbf{M}$. Let q be the maximum of $|\mathbf{A}|$ and 2^k and replace Φ with $\overline{\Phi}^q$. This does not change the value of r. Since $F(\mathbf{M})$ is distributive, we know that $F(\mathbf{M})$ is closed under connected primitive positive definitions. (The lattice obtained by adding 0 and 1 to $F(\mathbf{M})$ is closed under primitive positive definitions. As an interval in this lattice, $F(\mathbf{M})$ is closed under connected primitive positive definitions.) Therefore,

$$F(r) = F(\Phi(r_1,\dots,r_n)) = \Phi(F(r_1),\dots,F(r_n)) \in F(\mathbf{M}).$$

This means that there is some element of $\mathbf{M}$ which is δ related to r. Let $\overline{r}$ be the largest such element and $\underline{r}$ be the smallest.

Case 1: Assume first that $\underline{r} \neq 0_A$ and $\overline{r} \neq 1_A$. Notice then that the intervals $[0, \overline{r}]$ and $[\underline{r}, 1]$ as sublattices of $\mathbf{L}$ are closed under connected primitive positive definitions and are strictly smaller than $\mathbf{L}$. Now let $s = \Phi(r_1 \wedge \overline{r}, \ldots, r_n \wedge \overline{r})$ and $t = \Phi(r_1 \vee \underline{r}, \ldots, r_n \vee \underline{r})$. Then, $s \leq r \leq t$. We show that $s, t \in \mathbf{M}$. Let $\mathbf{N}$ be the interval $[0, \overline{r}]$ (in $\mathbf{L}$) as a lattice. Then $\mathbf{M} \cap \mathbf{N}$ is a $\delta_{\mathbf{N}}$-convex sublattice of $\mathbf{N}$ (since $\delta_{\mathbf{N}} \subseteq (\delta_{\mathbf{L}} \cap \mathbf{N}^2)$). By induction, then, we know that $\mathbf{M} \cap \mathbf{N}$ is closed under connected primitive positive definitions. Hence $s \in \mathbf{M} \cap \mathbf{N} \subseteq \mathbf{M}$. Similarly, $t \in \mathbf{M}$. We will show that $s, t \in r/\delta$. By δ-convexity, this will imply that $r \in \mathbf{M}$.

Notice that

$$\begin{aligned} F(s) &= F(\Phi(r_1 \wedge \overline{r}, \ldots, r_n \wedge \overline{r})) \\ &= \Phi(F(r_1) \wedge F(\overline{r}), \ldots, F(r_n) \wedge F(\overline{r})) \\ &= \Phi(F(r_1) \wedge F(r), \ldots, F(r_n) \wedge F(r)). \end{aligned}$$

However, since $F(\mathbf{M})$ is distributive, the lattice of equivalence relations on $\mathbf{2}^k$ obtained by adding the identity and universal relations (if necessary) is a power-hereditary congruence lattice, so by 2.4 there is a lattice term T so that this congruence lattice satisfies $T(x_1, \ldots, x_n) = \Phi(x_1, \ldots, x_n)$. Then we have

$$\begin{aligned} F(s) &= \Phi(F(r_1) \wedge F(r), \ldots, F(r_n) \wedge F(r)) \\ &= T(F(r_1) \wedge F(r), \ldots, F(r_n) \wedge F(r)) \\ &= T(F(r_1), \ldots, F(r_n)) \wedge F(r) \\ &= \Phi(F(r_1), \ldots, F(r_n)) \wedge F(r) \\ &= F(\Phi(r_1, \ldots, r_n)) \wedge F(r) \\ &= F(r) \wedge F(r) \\ &= F(r). \end{aligned}$$

The third equality follows from distributivity in $F(\mathbf{M})$. We note that here (along with the similar steps in following cases) is where the distributivity of $\mathbf{L}/\delta$ is essential. We can establish similarly that $F(t) = F(r)$. This means that $s, t \in r/\delta$. Since $s \leq r \leq t$ and $s, t \in \mathbf{M}$, δ-convexity now implies that $r \in \mathbf{M}$.

Case 2: Suppose now that $\underline{r} = 0_A$ and $\overline{r} = 1_A$. Then δ-convexity implies that $\mathbf{M} = \mathbf{L}$, so $r \in \mathbf{M}$.

Case 3: Suppose that $\underline{r} \neq 0_A$ but $\overline{r} = 1_A$. We have two subcases: either 1_A is join irreducible in $\mathbf{M}$ or it is not.

Subcase 3.1: Suppose that 1_A is join irreducible in $\mathbf{M}$. Let m be the unique subcover of 1_A in $\mathbf{M}$. If every path in the graph of $\Phi(r_1, \ldots, r_n)$ between the free variables has an edge labeled by 1_A, then $r = \Phi(r_1, \ldots, r_n) = 1_A = \overline{r} \in \mathbf{M}$. Assume this is not the case. Then we can remove from $\Phi(r_1,\ldots,r_n)$ any r_i which happens to equal 1_A without changing the value of $\Phi(r_1,\ldots,r_n)$. This means that each r_i can be assumed to be no greater than m. By applying the induction hypothesis to the interval $[0_A,m]$ (in $\mathbf{L}$), we can conclude that $r\in\mathbf{M}\cap[0_A,m]\subseteq\mathbf{M}$.

Subcase 3.2: Suppose that $a_1,\ldots,a_m\in\mathbf{M}$ with $a_1\vee\cdots\vee a_m=1_A$ so that no a_i is equal to 1. For each $i=1,\ldots,m$ let $t_i=\Phi(r_1\wedge a_i,\ldots,r_n\wedge a_i)$. Again by induction, we know that each $t_i\in\mathbf{M}\cap[0_A,a_i]\subseteq\mathbf{M}$. We also know that each $t_i\le r$ so that the join $t=t_1\vee\cdots\vee t_m$ is also below by r and is in $\mathbf{M}$. As in Case 1:

$$\begin{aligned}
F(t) &= F(\bigvee_{i=1}^{m}\Phi(r_1\wedge a_i,\ldots,r_n\wedge a_i))\\
&= \bigvee_{i=1}^{m}[\Phi(F(r_1)\wedge F(a_i),\ldots,F(r_n)\wedge F(a_i))]\\
&= \bigvee_{i=1}^{m}[\Phi(F(r_1),\ldots,F(r_n))\wedge F(a_i)]\\
&= \bigvee_{i=1}^{m}[F(\Phi(r_1,\ldots,r_n))\wedge F(a_i)]\\
&= \bigvee_{i=1}^{m}[F(r)\wedge F(a_i)]\\
&= F(r)\wedge\bigvee_{i=1}^{m}F(a_i)\\
&= F(r)\wedge F(\bigvee_{i=1}^{m}a_i)\\
&= F(r)\wedge 1_A\\
&= F(r).
\end{aligned}$$

Thus we have $t\delta r\delta\overline{r}=1_A$ and $t\le r\le 1_A$. Since $t,1_A\in\mathbf{M}$ we have $r\in\mathbf{M}$ by δ-convexity.

Case 4: Suppose now that $\underline{r}=0_A$ and that $\overline{r}\neq 1_A$. There are again two cases. Either 0_A is meet irreducible in $\mathbf{M}$ or it is not.

Subcase 4.1: Suppose that 0_A is meet irreducible in $\mathbf{M}$. Let m be the unique cover of 0_A in $\mathbf{M}$. If there is a path in the graph of $\Phi(r_1,\ldots,r_n)$ between the free variables labeled by 0_A, then $r=\Phi(r_1,\ldots,r_n)=0_A\in\mathbf{M}$. Assume that there is no such path. By identifying variables, we can remove from $\Phi(r_1,\ldots,r_n)$ all occurrences of 0_A without changing the value of $\Phi(r_1,\ldots,r_n)$. This means that every r_i can be assumed to be greater than or equal to m. By applying induction to the interval $[m,1_A]$, we can conclude that $r\in\mathbf{M}\cap[m,1_A]\subseteq\mathbf{M}$.

Subcase 4.2: Suppose that 0_A is not meet irreducible in $\mathbf{M}$. The argument for this case is dual to the argument in Subcase 3.1.

We now know by induction that every δ-convex 0-1 sublattice of Con$\mathbf{A}$ is closed under connected primitive positive definitions. This implies that such a sublattice is closed under all primitive positive definitions yielding equivalence relations. To see this, suppose that $\mathbf{M}$ is a 0-1 sublattice of Con$\mathbf{A}$ closed under connected primitive positive definitions, that Φ is a primitive positive formula, and that $r_1, \ldots, r_n \in \mathbf{M}$ with $r = \Phi(r_1, \ldots, r_n)$ an equivalence relation. If the free variables in $\Phi(r_1, \ldots, r_n)$ are not in the same component of the graph of $\Phi(r_1, \ldots, r_n)$, then $r = 1_A \in \mathbf{M}$. If the free variables in $\Phi(r_1, \ldots, r_n)$ are in the same component of this graph, then let Φ' be the primitive positive formula corresponding to the graph which is the component of the graph of $\Phi(r_1, \ldots, r_n)$ containing the free variables. Then Φ' is connected and $r = \Phi'(r_1, \ldots, r_n) \in \mathbf{M}$ (that is, r is completely determined by the connected component of the graph containing the free variables). □

Theorem 3.2 of [8] says that if $\mathbf{D} = \text{Con}\mathbf{A}$ is distributive and $\mathbf{M} = \text{Con}\mathbf{B}$, then every subdirect product of $\mathbf{D}$ and $\mathbf{M}$ is a congruence lattice on $A \times B$. This is a corollary of Theorem 4.6. Let $\mathbf{L}$ be a subdirect product of $\mathbf{D}$ and $\mathbf{M}$. Let η be the kernel of the projection of $\mathbf{D} \times \mathbf{M}$ onto $\mathbf{D}$. $\mathbf{L}$ is η-convex, and $\delta_{\mathbf{D}\times\mathbf{M}} \leq \eta$. Then $\mathbf{L}$ is $\delta_{\mathbf{D}\times\mathbf{M}}$-convex, and Theorem 4.6 applies.

If every equivalence class of $\delta_{\mathbf{L}}$ has at most two elements, then every sublattice of $\mathbf{L}$ is $\delta_{\mathbf{L}}$-convex, so Theorem 4.6 gives:

Corollary 4.7. *Suppose* $\mathbf{L}$ *is the congruence lattice of a finite algebra and that every* $\delta_{\mathbf{L}}$ *equivalence class has at most two elements. Then* $\mathbf{L}$ *is a hereditary congruence lattice.*

A natural application of Corollary 4.7 is to lattices obtained from distributive lattices by doublings. Alan Day's doubling construction for intervals in lattices was introduced in [2] to give a nonconstructive solution to the word problem for lattices. The construction was generalized to families of convex sets (rather than just intervals) in [4]. We are concerned only with the doubling of convex sets. Suppose that $\mathbf{L}$ is a lattice and that $C \subset \mathbf{L}$ is a convex subset. We define a lattice $\mathbf{L}[C]$ in the following way. The universe of $\mathbf{L}[C]$ is $(L - C) \cup (C \times 2)$. For any $x, y \in \mathbf{L}[C]$, we will define $x \leq y$ if one of these holds:

(1) $x, y \in L - C$ with $x \leq y$ in $\mathbf{L}$.
(2) $x = \langle a, i\rangle$, $y \in L - C$, and $a \leq y$ in $\mathbf{L}$.
(3) $x \in L - C$, $y = \langle b, i\rangle$, and $x \leq b$ in $\mathbf{L}$.
(4) $x = \langle a, i\rangle$ and $y = \langle b, j\rangle$ with $a \leq b$ and $i \leq j$.

It is not difficult to prove that $\leq$ is a lattice order on $\mathbf{L}[C]$ and that the function $\pi\colon \mathbf{L}[C] \to \mathbf{L}$ given by $\pi(x) = x$ for $x \in L - C$ and $\pi(\langle x, i\rangle) = x$ for $x \in C$ is a surjective lattice homomorphism. For any $x \in \mathbf{L}[C]$, if $x \in L - C$, then $x/\ker\pi$ has only one element. Otherwise, $x/\ker\pi$ has two elements. (This statement does not apply to the generalized doubling construction in [4].) Corollary 4.7 gives the following theorem.

Theorem 4.8. *Suppose* $\mathbf{L}$ *is a finite lattice obtained from a distributive lattice by doubling a convex set. Every congruence lattice representation of* $\mathbf{L}$ *is hereditary.*

5. Problems

There are several natural questions to ask in light of Corollary 4.7 and Theorem 4.8. Most of them are specific instances of this general question:

Problem 5.1. Which finite lattices $\mathbf{L}$ have the property that every representation of $\mathbf{L}$ as the congruence lattice of a finite algebra is (power-)hereditary?

Let $\mathcal{H}$ and $\mathcal{PH}$ be the classes of all finite lattices $\mathbf{L}$ so that every representation of $\mathbf{L}$ as the congruence lattice of a finite algebra is hereditary or power-hereditary. In Corollary 4.7, each equivalence class of $\delta_{\mathbf{L}}$ is either a one or two element lattice. Could these be replaced with arbitrary distributive lattices? Let $\mathcal{DIST}/\mathcal{DIST}$ be the class of all finite lattices $\mathbf{L}$ which have congruences δ so that $\mathbf{L}/\delta$ is distributive and every δ-class is a distributive lattice.

Problem 5.2. Which lattices in $\mathcal{DIST}/\mathcal{DIST}$ are in $\mathcal{H}$ (or $\mathcal{PH}$)?

For that matter:

Problem 5.3. Describe the lattices in $\mathcal{DIST}/\mathcal{DIST}$.

Problem 5.4. Can the word "hereditary" in 4.6, 4.7, or 4.8 be replaced with "power-hereditary?"

A finite lattice $\mathbf{L}$ is upper (lower) bounded if there is a surjective homomorphism f from a free lattice $\mathbf{F}$ onto $\mathbf{L}$ so that for every $b \in \mathbf{L}$ the set $f^{-1}(b)$ has a greatest (least) element. If $\mathbf{L}$ is both upper and lower bounded, then there exists such an f so that every $f^{-1}(b)$ has both a greatest and least element. In this case, $\mathbf{L}$ is called bounded.

A subset C of a lattice $\mathbf{L}$ is an upper pseudo-interval if C is a union of intervals which share the same top element. A lower pseudo-interval is defined dually. A lattice $\mathbf{L}$ is upper (lower) bounded if it can be obtained from a distributive lattice by a sequence of doublings of upper (lower) pseudo-intervals. A lattice $\mathbf{L}$ is bounded if and only if $\mathbf{L}$ can be obtained from a distributive lattice by a sequence of doublings of intervals [3]. It is natural to ask in 4.8 if the one doubling can be replaced with a sequence of doublings. This leads to:

Problem 5.5. Which (upper or lower) bounded lattices are in $\mathcal{H}$ (or $\mathcal{PH}$)?

It would also be interesting to investigate the classes $\mathcal{H}$ and $\mathcal{PH}$.

Problem 5.6. Under which of the operations of taking homomorphic images, sublattices, products, or subintervals are $\mathcal{H}$ and $\mathcal{PH}$ closed?

Finally, we would like to investigate lattices with the extreme opposite characteristics as those in $\mathcal{H}$ and $\mathcal{PH}$.

Problem 5.7. Is there a finite lattice $\mathbf{L}$ so that no representation of $\mathbf{L}$ as the congruence lattice of a finite algebra is hereditary?

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