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# A constructive approach to the finite congruence lattice representation problem

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*Abstract.* A finite lattice is **representable** if it is isomorphic to the congruence lattice of a finite algebra. In this paper, we develop methods by which we can construct new representable lattices from known ones. The techniques we employ are sufficient to show that every finite lattice which contains no three element antichains is representable. We then show that if an order polynomially complete lattice is representable then so is every one of its diagonal subdirect powers.

## 1. Introduction

One of the most elusive longstanding problems in Universal Algebra is the finite congruence representation problem: "Is every finite lattice isomorphic to the congruence lattice of a finite algebra?" Call a finite lattice which is isomorphic to the congruence lattice of a finite algebra **representable**. We begin here to construct a list of operations under which the class of representable lattices is closed. The motivation is quite simple. In [7], Pudlák and Tůma provide a construction on lattices so that any class of lattices which contains all Boolean lattices and is closed under their construction is the class of all lattices. This is then used to show that every finite lattice can be embedded in the lattice of equivalence relations on a finite set. A similar theorem which could be applied to the class of representable lattices would be the ultimate goal. More realistically, we might hope to develop operations by which we can construct new representable lattices from known ones, thereby enlarging the class of lattices known to be representable. The constructions covered herein are sufficient to prove that every finite lattice which contains no three element antichains is representable. We note that many of the tools we develop here, though unpublished, were known to P. Pudlák and J. Tůma.

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## 2. Primitive positive formulas

Call a subuniverse of a direct power of an algebra a **compatible relation** on the algebra. To begin with, we need to know when a system of relations on a finite set is the system of all finitary compatible relations of a finite algebra. There are many such characterizations, we choose to use the following model theoretic characterization.

Call a first order formula of the form $\exists\wedge$ (atomic) a **primitive positive** formula. Suppose $A$ is any set and $\sigma$ is a primitive positive formula using relation symbols $r_1, \ldots, r_n$. Let $r_1^A, \ldots, r_n^A$ be relations on $A$ so that $r_i$ has the same rank as $r_i^A$ for each $i$. Interpreting each $r_i$ in $\sigma$ as $r_i^A$ defines a relation $\sigma^A$ on the universe of $\mathbf{A}$. A set $\mathcal{R}$ of relations on $A$ is **closed under primitive positive definitions** if every relation $\sigma^A$ defined in this manner using only relations from $\mathcal{R}$ is already in $\mathcal{R}$. It is an easy exercise to show that the system of finitary compatible relations on an algebra is closed under primitive positive definitions. In the finite case, the reverse is also true:

LEMMA 2.1. [1, 5] *Suppose that $A$ is a finite set and $\mathcal{R}$ is a collection of finitary relations on $A$. There is an algebra $\mathbf{A}$ on $A$ for which $\mathcal{R}$ is the set of all finitary compatible relations if and only if $\mathcal{R}$ is closed under primitive positive definitions and contains $\emptyset$.*

We are concerned here with congruences, so we can isolate our attention to sets of equivalence relations. Lemma 2.1 has this immediate consequence about congruence lattices:

COROLLARY 2.2. *Suppose $\mathcal{L}$ is a 0–1 lattice of equivalence relations on a finite set $A$. There is an algebra $\mathbf{A}$ on $A$ with $\mathrm{Con}\mathbf{A} = \mathcal{L}$ if and only if every equivalence relation on $A$ which can be defined from $\mathcal{L}$ by a primitive positive formula is already in $\mathcal{L}$.*

Of course, a similar corollary would hold for any special type of compatible relation: tolerances, endomorphisms, homomorphisms between direct powers. It will be to our advantage to have a standard form for primitive positive formulas. This can be easily done. Since $\mathcal{L}$ in Corollary 2.2 must contain the identity and universal relations, we can always choose our primitive positive formulas to be of a special form:

COROLLARY 2.3. *Suppose $\mathcal{L}$ is a 0–1 lattice of equivalence relations on a finite set $A$. There is an algebra $\mathbf{A}$ on $A$ with $\mathrm{Con}\mathbf{A} = \mathcal{L}$ if and only if $\mathcal{L}$ contains every equivalence relation on $A$ definable by primitive positive formulas of the form*

$$\sigma(x_1, x_2) \longleftrightarrow \exists x_3, \ldots, x_n \bigwedge_{i,j \leq n} S_{i,j}(x_i, x_j)$$

*where each $s_{i,j} \in \mathcal{L}$.*

We note here that equivalence relations have many faces. They are sets, but they are also relations. As such, sometimes it is convenient to use set notation to denote inclusion– $\langle x, y\rangle \in \alpha$ – and other times it is useful to use more model theoretic notation – $x\alpha y$ or $\alpha(x, y)$. Lattices also have more than one face. Sometimes, we will be working with lattices of equivalence relations. In these cases, we will use scripted text – $\mathcal{L}$ – to refer to the lattices. Whenever we are working with a lattice which is not necessarily a lattice of equivalence relations, we will use bold faced text – **L**.

## 3. The tools

There is an intimate relationship between primitive positive formulas using only binary relation symbols and certain graphs. This relationship is essential to our first two lemmas. Suppose $\sigma$ is a primitive positive formula containing the variables $x_1, \ldots, x_n$. By the **graph of** $\sigma$ we mean the graph **G** whose vertices are labeled $x_1, \ldots, x_n$ so that for each $i, j \leq n$ there is an edge between $x_i$ and $x_j$ labeled by $r$ if and only if $r(x_i, x_j)$ is included in the conjunction in $\sigma$. For example, if $\sigma$ is defined by

$$\sigma(x_1, x_2) \longleftrightarrow \exists x_3, x_4[r(x_1, x_3) \wedge s(x_1, x_4) \wedge t(x_3, x_4) \wedge u(x_3, x_2) \wedge v(x_4, x_2)]$$

then the graph of $\sigma$ would look like the graph in Figure 1.

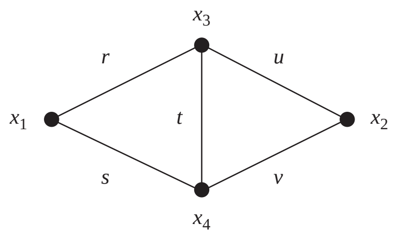


Figure 1 The graph of $\sigma$

Suppose $\sigma(x_1, x_2)$ is a primitive positive formula involving binary relation symbols $r_1, \ldots, r_n$ and that **G** is the graph of $\sigma$. Suppose $A$ is a finite set and that $r_1^A, \ldots, r_n^A$ are equivalence relations on $A$. Let $\sigma^A$ be the relation on $A$ obtained by interpretting each $r_i$ as $r_i^A$. In the language of [9], $\sigma^A$ is the relation obtained from the graphical composition of $r_1^A, \ldots, r_n^A$ and **G** associated with the labelling of the edge $r_i$ in **G** by $r_i^A$ ([9] uses directed graphs, but this is not necessary if we are only using equivalence relations). Furthermore, every such graphical composition can be realized through a primitive positive formula. Thus Corollary 2.2 is equivalent to Theorem 2.6 of [9] which says that the lattices of equivalence relations on a finite set are those lattices of equivalence relations which are closed under all graphical compositions.

LEMMA 3.1. *Suppose $\alpha \leq \beta$ are equivalence relations on a finite set $A$. Let $\sigma$ be a binary relation on $A$ defined by*

$$\sigma(x_1, x_2) \longleftrightarrow \exists x_3, \ldots, x_n \bigwedge_{i,j,\leq n} s_{i,j}(x_i, x_j)$$

*where each $s_{i,j} \in \{\alpha, \beta\}$. Then $\sigma \in \{\alpha, \beta\}$ and $\sigma = \alpha$ if and only if there is a path of $\alpha$-edges from $x_1$ to $x_2$ in the graph of $\sigma$.*

*Proof.* The lemma is trivial if $\alpha = \beta$, so assume $\alpha < \beta$. It should be clear that $\alpha \leq \sigma \leq \beta$. If there is a path labeled by $\alpha$ connecting $x_1$ and $x_2$ in the graph of $\sigma$, then it follows immediately that $\sigma(x_1, x_2)$ implies $\alpha(x_1, x_2)$. Thus we would have $\sigma \leq \alpha$, and hence $\sigma = \alpha$. Assume then that there is no such path. We will show that $\sigma = \beta$. This will establish the lemma. We know already that $\sigma \leq \beta$. We show that $\beta \leq \sigma$. Suppose $a, b \in A$ and $\beta(a, b)$. Let **G** be the graph of $\sigma$ with all of the $\beta$-edges removed. For $i = 1, \ldots, n$ define:

$$y_i = \begin{cases} a & \text{if } x_i \text{ is in the } \mathbf{G}\text{-component of } x_1 \\ b & \text{else} \end{cases}$$

For all $i, j \leq n$, we claim that $s_{i,j}(y_i, y_j)$ holds. Since everything here is $\beta$ related, we just need to show that $s_{i,j}(y_i, y_j)$ holds when $s_{i,j} = \alpha$. When $s_{i,j} = \alpha$ the vertices $x_i$ and $x_j$ are in the same component of **G**, so by definition $y_i = y_j$. Hence $s_{i,j}(y_i, y_j)$ is trivial. Since $\bigwedge_{i,j \leq n} s_{i,j}(y_i, y_j)$ holds, it follows that $\sigma(y_1, y_2)$. But $y_1 = a$ and $y_2 = b$, so $\sigma(a, b)$ holds. Thus $\beta \leq \sigma$, so actually $\beta = \sigma$. □

In this lemma, the fact that $\sigma \in \{\alpha, \beta\}$ can easily be seen by other methods. What is essential for us is the conclusion regarding paths in the graph of $\sigma$. It is vital for the next lemma.

LEMMA 3.2. *Suppose* **A** *is a finite algebra and $\alpha$ and $\beta$ are equivalence relations on $A$. There is an algebra* $\mathbf{A}'$ *on* **A** *with*

$$\text{Con}\mathbf{A}' = \{x \in \text{Con}\mathbf{A} : x \leq \alpha \textit{ or } x \geq \beta\}.$$

*Proof.* If $\alpha = 1_A$ or $\beta = 0_A$, then this is trivial. Assume then that $\alpha < 1_A$ and $\beta > 0_A$. Let $\mathcal{L} = \{x \in \text{Con}\mathbf{A} : x \leq \alpha \text{ or } x \geq \beta\}$. We will show that $\mathcal{L}$ is closed under primitive positive definitions which yield equivalence relations on $A$. Define a binary relation $\rho$ on $A$ by

$$\rho(x_1, x_2) \longleftrightarrow \exists x_3, \ldots, x_n \bigwedge_{i,j,\leq n} r_{i,j}(x_i, x_j)$$

where each $r_{i,j} \in \mathcal{L}$. Suppose that $\rho$ is an equivalence relation on $A$. We need to show that $\rho \in \mathcal{L}$. Note that $\rho \in \mathrm{Con}\mathbf{A}$ since $\mathrm{Con}\mathbf{A}$ is closed under the appropriate primitive positive defintions.

For each $i, j \leq n$ define:

$$s_{i,j} = \begin{cases} \alpha & r_{i,j} \leq \alpha \\ 1_A & \text{else} \end{cases} \quad \text{and} \quad t_{i,j} = \begin{cases} \beta & r_{i,j} \leq \beta \\ 0_A & \text{else} \end{cases}$$

Also define:

$$\sigma(x_1, x_2) \longleftrightarrow \exists x_3, \ldots, x_n \bigwedge_{i,j,\leq n} s_{i,j}(x_i, x_j)$$

and

$$\tau(x_1, x_2) \longleftrightarrow \exists x_3, \ldots, x_n \bigwedge_{i,j \leq n} t_{i,j}(x_i, x_j).$$

Since $t_{i,j} \leq r_{i,j} \leq s_{i,j}$ for all $i, j \leq n$, we have $\tau \leq \rho \leq \sigma$. We will show that either $\tau = \beta$ or $\sigma = \alpha$. Since $\rho \in \mathrm{Con}\mathbf{A}$, this will establish that $\rho \in \mathcal{L}$ as desired. From Lemma 3.1, we know that $\tau \in \{0_A, \beta\}$. Assume that $\tau \neq \beta$. From the lemma, there is a path in the graph of $\tau$ connecting $x_1$ and $x_2$ with every edge labeled by $0_A$. Therefore, there is a path in the graph of $\rho$ from $x_1$ to $x_2$ with every edge labeled by an equivalence relation below $\alpha$ (since every equivalence relation in $\mathcal{L}$ is either below $\alpha$ or above $\beta$). It follows that in the graph of $\sigma$ there is a path connecting $x_1$ and $x_2$ whose edges are labeled with $\alpha$. From Lemma 3.1, we can conclude that $\sigma = \alpha$. □

The next lemma is almost obvious:

LEMMA 3.3. *Suppose* **A** *and* **B** *are algebras on a set* $A$. *There is an algebra* **C** *on* $A$ *so that* $\mathrm{Con}\mathbf{C} = \mathrm{Con}\mathbf{A} \cap \mathrm{Con}\mathbf{B}$. □

We will make extensive use of Lemma 3.2 a little later. However, we first concern ourselves with the fact that the class of finite representable lattices is closed under subintervals.

LEMMA 3.4. *The class of representable lattices is closed under subintervals.*

*Proof.* Suppose **L** is a finite representable lattice. Find a finite algebra **A** so that $\mathrm{Con}\mathbf{A} \cong \mathbf{L}$. Suppose $a < b$ in $\mathrm{Con}\mathbf{A}$. Let $\mathbf{C} = \mathbf{A}/a$ and $\theta = b/a$. The interval $[0_C, \theta]$ is isomorphic as a lattice to the interval $[a, b]$. If $\theta = 1_C$, then we are done. Assume, therefore, that $\theta < 1_C$, and let $H_1, \ldots, H_m$ be the equivalence classes of $\theta$. For any binary relation $\sigma \subseteq \theta$, let $\sigma^i = \sigma \cap H_i^2$ (that is, $\sigma^i$ is the restriction of $\sigma$ to $H_i$). Let $B = \prod_{i=1}^m H_i$, and define $F : [0_C, \theta] \to \mathrm{Eq}(B)$ by $F(\sigma) = \prod_{i=1}^m \sigma^i$. Let $\mathcal{N} = F([0_C, \theta])$.

Notice first that for any equivalence relations $\sigma, \tau \leq \theta$ on $C$ we have $\sigma \leq \tau$ if and only if $\sigma^i \leq \tau^i$ for all $i$. It follows that $F$ is a lattice injection, and $\mathcal{N}$ is isomorphic as a lattice to the interval $[a, b]$. Also notice that $\mathcal{N}$ is a 0–1 lattice of equivalence relations on $B$. Hence we need only show that there is an algebra $\mathbf{B}$ on $B$ with Con$\mathbf{B} = \mathcal{N}$. To do this we show that $\mathcal{N}$ is closed under all primitive positive definitions which happen to yield equivalence relations on $B$. Suppose $\rho$ is a binary relation on $B$ given by:

$$\rho(x_1, x_2) \longleftrightarrow \exists x_3, \dots, x_n \bigwedge_{i,j,\leq n} r_{i,j}(x_i, x_j)$$

where each $r_{i,j} \in \mathcal{N}$. For each $i$, $j \leq n$, we can find $s_{i,j} \leq \theta$ in Con$\mathbf{C}$ so that

$$r_{i,j} = F(s_{i,j}) = \prod_{k=1}^{m} s_{i,j}^k.$$

For each $k = 1, \dots, m$, define $t^k$ on $H_k$ by:

$$\tau^k(x_1, x_2) \longleftrightarrow \exists x_3, \dots, x_n \bigwedge_{i,j \leq n} s_{i,j}^k(x_i, x_j).$$

Also, define $\sigma$ on $C$ by

$$\sigma(x_1, x_2) \longleftrightarrow \exists x_3, \dots, x_n \bigwedge_{i,j \leq n} s_{i,j}(x_i, x_j).$$

It is not difficult to see that $\tau^k = \sigma^k$ for each $k$ since all of the relations present in the definition of $\sigma$ must hold within individual $\theta$ classes. Also, $\rho = \prod_{k=1}^{m} \tau^k$ is not hard to see. Thus

$$\rho = \prod_{k=1}^{m} \tau^k = \prod_{k=1}^{m} \sigma^k.$$

Suppose $\rho$ is an equivalence relation. Since $\rho = \prod_{k=1}^{m} \sigma^k$, it follows that each $\sigma^k$ is also an equivalence relation. It is routine to show that $\sigma$ is an equivalence relation – since in each case of reflexivity, symmetry, and transitivity we can isolate our attention to a single equivalence class of $\theta$ and thus work with one of the $\sigma^k$. Thus $\sigma$ is an equivalence relation on $\mathbf{C}$ defined using primitive positive definitions involving only members of Con$\mathbf{C}$ less than $\theta$, so $\sigma \in$ Con$\mathbf{C}$ and $\sigma \leq \theta$. Hence $\rho = \prod_{k=1}^{m} \sigma^k = F(\sigma)$ is a member of $\mathcal{N}$. We have established that $\mathcal{N}$ is closed under primitive positive definitions yielding equivalence relations. Since $\mathcal{N}$ is isomorphic as a lattice to the original interval $[a, b]$, this interval is representable. $\square$

It follows from a result of Kurzweil [3] that the class of finite representable lattices is self-dual. This fact provides a much simpler (though not self-contained) proof of Lemma 3.4. We choose to state our proof because we believe that the connection used herein between equivalence relations on $\cup_{k=1}^{m} H_k$ and equivalence relations on $\prod_{k=1}^{m} H_k$ might prove useful in attempting to show that the class of representable lattices is closed under subdirect products. Notice that the interval $[0_C, \theta]$ in the proof induces through restriction a lattice of equivalence relations on each $H_k$. The representation which is found for this interval in the proof is actually a subdirect product of these induced lattices. We will return to subdirect products in Section 5.

Since the congruence lattice of the non-indexed product of two algebras **A** and **B** is isomorphic to Con**A** $\times$ Con**B**, and since any two lattices **L** and **N** can be embedded as intervals in the product **L** $\times$ **N**, we immediately have:

LEMMA 3.5. *Suppose* **L** *and* **N** *are finite lattices.* **L** $\times$ **N** *is representable if and only if both* **L** *and* **N** *are representable.* □

We next tackle ordinal sums and things that look like ordinal sums. Suppose **L** and **N** are lattices. We will denote the lattice obtained from the ordinal sum **L** $\oplus$ **N** by identifying the least element of the upper lattice with the greatest element of the lower by **L** $\oplus_a$ **N** (see Figure 2).

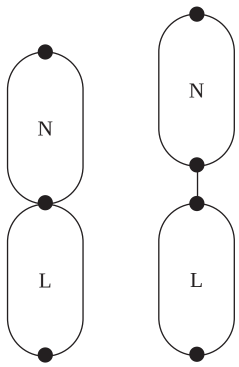


Figure 2 **L**$\oplus_a$ **N** (left) and **L**$\oplus$ **N** (right).

LEMMA 3.6. *Suppose* **L** *and* **N** *are finite lattices.* **L** $\oplus_a$ **N** *is representable if and only if* **L** *and* **N** *are representable.*

*Proof.* If **L** $\oplus_a$ **N** is representable, then **L** and **N** are representable by Lemma 3.4 (as subintervals). Suppose **L** and **N** are representable. Notice that **L** $\oplus_a$ **N** is isomorphic to the sublattice of (the representable lattice) $\mathbf{L} \times \mathbf{N}$ containing the elements

$$\{x \in \mathbf{L} \times \mathbf{N} : x \leq \langle 1, 0 \rangle \text{ or } x \geq \langle 1, 0 \rangle\}.$$

By lemma 3.2, this sublattice is also representable. □

COROLLARY 3.7. *Suppose* $\mathbf{L}$ *and* $\mathbf{N}$ *are finite lattices.* $\mathbf{L} \oplus \mathbf{N}$ *is representable if and only if* $\mathbf{L}$ *and* $\mathbf{N}$ *are representable.*

*Proof.* If $\mathbf{L} \oplus \mathbf{N}$ is representable, then $\mathbf{L}$ and $\mathbf{N}$ are representable by Lemma 3.4. Let $\mathbf{2}$ be the two element lattice (which is representable). Suppose $\mathbf{L}$ and $\mathbf{N}$ representable. Since $\mathbf{L} \oplus \mathbf{N} \cong (\mathbf{L} \oplus_a \mathbf{2}) \oplus_a \mathbf{N}$, it follows that $\mathbf{L} \oplus \mathbf{N}$ is representable by Lemma 3.6. □

Suppose $\mathbf{L}$ and $\mathbf{N}$ are lattices. We define a new lattice $\mathbf{L} \mp \mathbf{N}$ which we call the **parallel sum of L and N**. The universe of $\mathbf{L} \mp \mathbf{N}$ is the disjoint union of $\mathbf{L}$ and $\mathbf{N}$ along with two new elements which we will denote here as 0 and 1. The order on $\mathbf{L} \mp \mathbf{N}$ is given by:

$$x \leq y \longleftrightarrow \begin{cases} x \leq y \in \mathbf{L} \\ x \leq y \in \mathbf{N} \\ y = 1 \\ x = 0 \end{cases}$$

(See Figure 3).

There is a rather technical extension of Lemma 3.2 which we will employ in Lemmas 3.9 and 3.10. This corollary follows immediately from Lemma 3.2 and Lemma 3.3.

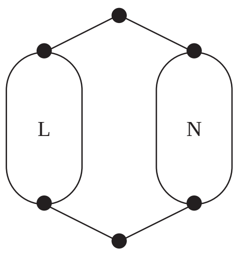


Figure 3 $\mathbf{N} \mp \mathbf{N}$

COROLLARY 3.8. *Suppose* $\mathbf{A}$ *is a finite algebra and* $\alpha \leq \beta, \gamma \leq \delta$ *are congruences on* $\mathbf{A}$. *If* $\alpha \vee \gamma = 1_A$ *and* $\beta \wedge \delta = 0_A$, *then there is an algebra* $\mathbf{A}'$ *on* $A$ *with* $\text{Con}\mathbf{A}' = \{0_A\} \cup [\alpha, \beta] \cup [\gamma, \delta] \cup \{1_a\}$.

We will denote the trivial lattice by $\mathbf{1}$.

LEMMA 3.9. *If* $\mathbf{L}$ *is representable, then* $\mathbf{L} \mp \mathbf{1}$ *is representable.*

*Proof.* Let $\mathbf{A}$ be a finite algebra so that Con $\mathbf{A} \cong \mathbf{L}$. We can assume that $\mathbf{A}$ is unary. List the elements of $A$ as $A = \{r_1, \ldots, r_n\}$. Let $C = \{l_1, \ldots, l_n\}$, $B = C \cup A$, and define three equivalence relations, $\alpha$, $\beta$, and $\gamma$ on $B$ so that: $\alpha = C^2 \cup \Delta_A$, $\beta = C^2 \cup A^2$, and $\gamma$ identifies $l_i$ and $r_i$ for each $i$ and nothing else. Note that $\beta \wedge \gamma = 0_B$ and $\alpha \vee \gamma = 1_B$. Since the entire lattice of equivalence relations on $B$ is the congruence lattice of an algebra on $B$, Corollary 3.8 now tells us that there is an algebra $\mathbf{B}_1$ on $B$ with Con$\mathbf{B}_1 = \{0_B, 1_B\} \cup [\alpha, \beta] \cup \{\gamma\}$. There is a lattice isomorphism of the lattice of equivalence relations on $A$ and the interval $[\alpha, \beta]$ given by $\theta \mapsto \bar{\theta}$ where $\bar{\theta} = \theta \cup C^2$.

We can extend any operation $p$ of $\mathbf{A}$ to an operation $\bar{p}$ on $B$ by defining $\bar{p}(r_i) = p(r_i)$ and $\bar{p}(l_i) = l_j$ if and only if $p(r_i) = r_j$. Let $F$ be the set of all $\bar{p}$ where $p$ is a basic operation of $A$.

Define $\mathbf{B}$ to be the algebra on $B$ with the operations of $\mathbf{B}_1$ along with the operations in $F$. Then Con$\mathbf{B}$ is contained in Con$\mathbf{B}_1$. We first notice that $\gamma \in$ Con$\mathbf{B}$. To do so, we simply need to establish that $\gamma$ is closed under the operations in $F$. Let $\bar{p} \in F$. Suppose $\langle x, y\rangle \in \gamma$. If $x = y$, then it is trivial that $\langle \bar{p}(x), \bar{p}(y)\rangle \in \gamma$. If $x \neq y$, then we can assume that $x = l_i$ and $y = r_i$ for some $i$. For some $j$, it must be that $\langle \bar{p}(x), \bar{p}(y)\rangle = \langle l_j, r_j\rangle \in \gamma$. Thus $\gamma$ is closed under the operations of $F$ and, hence under all of the operations of $\mathbf{B}$.

We claim that for any equivalence relation $\bar{\theta}$ in $[\alpha, \beta]$, $\bar{\theta} \in$ Con$\mathbf{B}$ if and only if $\theta \in$ Con$\mathbf{A}$. Suppose first that $\theta \in$ Con$\mathbf{A}$. We show that $\bar{\theta}$ is preserved by the operations of $\mathbf{B}$. Again, we need only concern ourselves with the operations in $F$. Suppose $\bar{p} \in F$ and $\langle x, y\rangle \in \bar{\theta}$. Either $\langle x, y\rangle \in C^2$ or $\langle x, y\rangle \in \theta$. If $\langle x, y\rangle \in C^2$, then $\langle \bar{p}(x), \bar{p}(y)\rangle$ is contained in $C^2 \subseteq \bar{\theta}$ since $C$ is closed under the operations in $F$. If $\langle x, y\rangle \in \theta$ then $\langle \bar{p}(x), \bar{p}(y)\rangle = \langle p(x), p(y)\rangle \in \theta \subseteq \bar{\theta}$. This shows that if $\theta \in$ Con$\mathbf{A}$ then $\bar{\theta} \in$ Con$\mathbf{B}$. To establish the reverse implication, we use the contrapositive. Suppose $\theta \notin$ Con$\mathbf{A}$. There is some operation $p$ of $\mathbf{A}$ so that $\theta$ is not closed under $p$. It follows immediately that $\bar{\theta}$ is not closed under $\bar{p}$, and so $\bar{\theta}$ is not in Con$\mathbf{B}$. We have that

$$\mathrm{Con}\mathbf{B} = \{0_B, 1_B, \gamma\} \cup \{\bar{\theta} : \theta \in \mathrm{Con}\mathbf{A}\}.$$

It is easy to check that this is isomorphic to $\mathbf{L} \mp \mathbf{1}$. □

LEMMA 3.10. *Suppose* $\mathbf{L}_1$ *and* $\mathbf{L}_2$ *are finite lattices.* $\mathbf{L}_1 \mp \mathbf{L}_2$ *is representable if and only if* $\mathbf{L}_1$ *and* $\mathbf{L}_2$ *are representable.*

*Proof.* If $\mathbf{L}_1 \mp \mathbf{L}_2$ is representable, then both of $\mathbf{L}_1$ and $\mathbf{L}_2$ are by Lemma 3.4. Suppose that $\mathbf{L}_1$ and $\mathbf{L}_2$ are both representable. By the previous lemma, $\mathbf{L}_1 \mp \mathbf{1}$ and $\mathbf{L}_2 \mp \mathbf{1}$ are representable. Let $a_i$ be the point incomparable to $\mathbf{L}_i$ in $\mathbf{L}_i \mp \mathbf{1}$. Let $1_i$ and $0_i$ be the greatest and least elements of $\mathbf{L}_i$. Within $(\mathbf{L}_1 \mp \mathbf{1}) \times (\mathbf{L}_2 \mp \mathbf{1})$

$$\langle 1_1, a_2\rangle \wedge \langle a_1, 1_2\rangle = \langle 0_1, 0_2\rangle \text{ and } \langle 0_1, a_2\rangle \vee \langle a_1, 0_2\rangle = \langle 1_1, 1_2\rangle.$$

By Corollary 3.8, the sublattice of $(\mathbf{L}_1 \mp \mathbf{1}) \times (\mathbf{L}_2 \mp \mathbf{1})$ consisting of the elements

$$\{\langle 1_1, 1_2\rangle, \langle 0_1, 0_2\rangle\} \cup [\langle 0_1, a_2\rangle, \langle 1_1, a_2\rangle] \cup [\langle a_1, 0_2\rangle, \langle a_1, 1_2\rangle]$$

is representable. It is not difficult to see that this sublattice is isomorphic to $\mathbf{L}_1 \mp \mathbf{L}_2$. □

## 4. An example

In this section, we apply the tools we have developed so far to show that any finite lattice which contains no three element antichains is representable. This provides an example of how the tools we have developed can be applied, but we note that the techniques of [6] can also be used to prove this result. We also note that every lattice which contains no three element antichains can be embedded into a subdirect power of $\mathbf{N}_5$ [4].

Suppose $s$ and $t$ are positive integers. By $\mathbf{N}_{s,t}$ we mean the parallel sum of an $s$-element chain with a $t$-element chain. Since every finite distributive lattice is representable [8], Lemma 3.10 immediately gives us:

LEMMA 4.1. *Every* $\mathbf{N}_{s,t}$ *is representable.*

We are now ready for:

THEOREM 4.2. *Every finite lattice which contains no three element antichains is representable.*

*Proof.* Let $\mathbf{L}$ be a finite lattice with no three-element antichains. We show by induction on the size of $\mathbf{L}$ that $\mathbf{L}$ is representable. The reader can find examples to show that every lattice with fewer than 6 elements is representable. Therefore, suppose $|\mathbf{L}| \geq 6$ and that every lattice smaller than $\mathbf{L}$ which does not contain a three element antichain is representable.

Suppose first that $\mathbf{L}$ has a single atom $a$. Let $\mathbf{L}'$ be the interval in $\mathbf{L}$ above $a$. $\mathbf{L}'$ is clearly smaller than $\mathbf{L}$ and has no three element antichains. By induction $\mathbf{L}'$ is representable. By Lemma 3.6, $\mathbf{L}$ is representable since $\mathbf{L} \cong \mathbf{2} \oplus_a \mathbf{L}'$.

Next, suppose that $\mathbf{L}$ has more than one atom. Since $\mathbf{L}$ cannot have a three element antichain, $\mathbf{L}$ has precisely two atoms $a$ and $b$. If $a \vee b = 1$, then $\mathbf{L}$ is isomorphic to some $\mathbf{N}_{s,t}$ and is representable by Lemma 4.1. Suppose then that $c = a \vee b < 1$. There are two cases – either everything in $\mathbf{L}$ is comparable to $c$ or there is something incomparable to $c$.

Suppose first that everything in $\mathbf{L}$ is comparable to $c$. Let $\mathbf{L}_1$ be the lattice of elements greater than or equal to $c$ and let $\mathbf{L}_2$ be the lattice of elements less than or equal to $c$. Each of $\mathbf{L}_1$ and $\mathbf{L}_2$ is smaller than $\mathbf{L}$, and neither contains a three element antichain. By induction, both are representable. Since $\mathbf{L} \cong \mathbf{L}_2 \oplus_a \mathbf{L}_1$, $\mathbf{L}$ is also representable by Lemma 3.6.

Suppose now that there are elements of $\mathbf{L}$ incomparable to $c$. Since $\mathbf{L}$ contains no three element antichains, the set of elements incomparable to $c$ must form a chain. By finiteness, we can select the greatest element $d$ incomparable to $c$. The element $d$ must exceed either $a$ or $b$ (not both since $c = a \vee b$). Without loss of generality, assume that $d > a$. Let $a' = d \wedge c$. Notice that $c$ must cover $a'$ since any element strictly between $a'$ and $c$ must be incomparable both to $d$ and $b$ – and since $\mathbf{L}$ contains no three element antichains. Our lattice $\mathbf{L}$ now looks like the lattice in Figure 4.

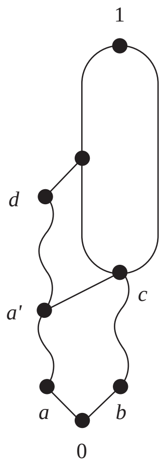


Figure 4 The latice $\mathbf{L}$ in Theorem 4.2.

Let $\mathbf{L}_1 = [a', 1]$, and let $\mathbf{L}_2 = [0, c]$. By induction again each of these lattices is representable, so the lattice $\mathbf{L}_1 \times \mathbf{L}_2$ is representable by Lemma 3.5. Define $f : \mathbf{L} \to \mathbf{L}_1 \times \mathbf{L}_2$ by $f(x) = \langle x \vee a', x \wedge c \rangle$. It is routine to check that $f$ is a lattice injection. Therefore, the image

$$f(\mathbf{L}) = ([a', d] \times \{a\}) \cup ([c, 1] \times \{c\}) \cup (\{a'\} \times [0, a']) \cup (\{c\} \times [b, c])$$

is isomorphic to $\mathbf{L}$. If

$$\mathbf{M} = \{x \in \mathbf{L}_1 \times \mathbf{L}_2 : x \geq \langle a', a' \rangle \text{ or } x \leq \langle c, c \rangle\},$$

then

$$f(\mathbf{L}) = \{x \in \mathbf{M} : x \geq \langle c, b \rangle \text{ or } r \leq \langle d, a' \rangle\}.$$

Since $\mathbf{L}_1 \times \mathbf{L}_2$ is representable, it follows from Lemma 3.2 that $\mathbf{M}$, $f(\mathbf{L})$, and hence $\mathbf{L}$ are all representable. ☐

## 5. OPC lattices and subdirect powers

A lattice is **order polynomially complete** (or **OPC**) if every order preserving operation on the lattice is a polynomial of the lattice. While OPC lattices are special, they arise naturally when considering the finite congruence representation problem. Every finite simple lattice whose atoms join to the maximal element is OPC [10]. This includes all equivalence relation lattices on finite sets, and it includes all of the lattices $\mathbf{M}_n$ (which many believe will hold special significance in the solution to this problem [7]).

Suppose $\sigma(x_1, x_2)$ is a primitive positive formula with two free variables which involves only the relation symbols $r_1, \ldots, r_n$ – all binary. Suppose also that $A$ is any finite set. We can think of $\sigma$ as defining an $n$-ary operation on the set of binary relations of $A$. If $r_1^A, \ldots, r_n^A$ are binary relations on $A$, then the operation would map $\langle r_1^A, \ldots, r_n^A \rangle$ to the relation defined on $A$ when each $r_i$ in $\sigma$ is interpreted as $r_i^A$. We will denote this relation as $\sigma(r_1^A, \ldots, r_n^A)$. If $\sigma$ is such that $\sigma(r_1^A, \ldots, r_n^A)$ is an equivalence relation when each $r_i^A$ is, then $\sigma$ actually defines an operation on $\mathrm{Eq}(A)$ (the lattice of equivalence relations on $A$). If $\mathbf{G}$ is the graph of $\sigma$, then this operation is essentially the graphical composition $P_{\mathbf{G},x_1,x_2}$ of [9]. Since this operation must necessarily be order preserving, and since $\mathrm{Eq}(A)$ is OPC, the operation defined by $\sigma$ could be realized as a polynomial of the lattice $\mathrm{Eq}(A)$. This observation is the basis of:

THEOREM 5.1. *Suppose $A$ is a finite set. There is a set $P$ of polynomials of* $\langle \mathrm{Eq}(A), \wedge, \vee \rangle$ *so that any* 0–1 *lattice $\mathcal{L}$ of equivalence relations on $A$ is the congruence lattice of an algebra on $A$ if and only if $\mathcal{L}$ is closed under the operations in $P$.*

*Proof.* Suppose $A$ is a finite set. Let $\sigma$ be any primitive positive formula with two free variables involving the relation symbols $r_1, \ldots, r_n$. If $r_1^A, \ldots, r_n^A$ are equivalence relations on $A$, then $\sigma(r_1^A, \ldots, r_n^A)$ is necessarily reflexive. Define $\sigma'$ by

$$\sigma'(x_1, x_2) \longleftrightarrow \sigma(x_1, x_2) \wedge \sigma(x_2, x_1).$$

Then $\sigma'$ is (equivalent to) a primitive positive formula and involves only the binary relation symbols $r_1, \ldots, r_n$. If $r_1^A, \ldots, r_n^A$ are equivalence relations on $A$, then $\sigma'(r_1^A, \ldots, r_n^A)$ is reflexive and symmetric. Define $\hat{\sigma}$ by

$$\hat{\sigma}(x_1, x_{|A|}) \longleftrightarrow \exists x_2, \ldots, x_{|A|-1} \bigwedge_{i=1}^{|A|-1} \sigma'(x_i, x_{i+1}).$$

Then $\hat{\sigma}$ is (equivalent to) a primitive positive formula and involves only the binary relation symbols $r_1, \ldots, r_n$. If $r_1^A, \ldots, r_n^A$ are equivalence relations on $A$, then $\hat{\sigma}(r_1^A, \ldots, r_n^A)$ is an equivalence relation. Moreover, if $\sigma(r_1^A, \ldots, r_n^A)$ is an equivalence relation, then $\hat{\sigma}(r_1^A, \ldots, r_n^A)$ and $\sigma(r_1^A, \ldots, r_n^A)$ are equal. The operation on $\mathrm{Eq}(A)$ defined by $\hat{\sigma}$ is order preserving, so we can find an $n$-ary polynomial $p_\sigma$ of $\langle \mathrm{Eq}(A), \wedge, \vee\rangle$ so that $p_\sigma(r_1^A, \ldots, r_n^A) = \hat{\sigma}(r_1^A, \ldots, r_n^A)$ for any equivalence relations $r_1^A, \ldots, r_n^A$ on $A$. Let $P$ be the set of all such $p_\sigma$ for every binary primitive positive formula $\sigma$ involving only binary relation symbols. It should now be clear that a lattice of equivalence on $A$ is closed under primitive positive definitions if and only if it is closed under the operations in $P$. □

We can use the idea of interpolating the operation on an equivalence relation lattice defined by a primitive positive formula to approach the representability of diagonal subdirect powers of OPC lattices. A **diagonal subdirect power** of an algebra **A** is a subdirect power which contains the diagonal relation.

THEOREM 5.2. *Suppose* **L** *is a finite representable lattice. If* **L** *is OPC, then every diagonal subdirect power of* **L** *is also representable.*

*Proof.* Let **A** be a finite algebra for which there is an isomorphism $f : \mathbf{L} \to \mathrm{Con}\mathbf{A}$. Suppose **M** is a diagonal sublattice of $\mathbf{L}^n$. Within $\mathrm{Eq}(A^n)$, consider the lattice $\mathcal{N}$ of equivalence relations of the form $\prod_{i=1}^n \theta_i$ where each $\theta_i \in \mathrm{Con}\mathbf{A}$. $\mathcal{N}$ is closed under primitive positive definitions which yield equivalence relations (it is congruence lattice of the non-indexed product of $n$ copies of **A**). We will view the members of $\mathcal{N}$ as $n$-tuples of elements from Con**A**. Allowing $f$ to act coordinate-wise on $\mathbf{L}^n$ gives an isomorphism of $\mathbf{L}^n$ with $\mathcal{N}$. The restriction of this isomorphism to **M** gives an isomorphic copy $\mathcal{M}$ of **M** in $\mathcal{N}$ which contains the diagonal relation. We will show that $\mathcal{M}$ is closed under all primitive positive definitions which yield equivalence relations.

Suppose $\sigma$ is a primitive positive formula involving the binary relation symbols $r_1, \ldots, r_m$. Suppose $\theta^1, \ldots, \theta^m$ are members of $\mathcal{M}$. We will assume $\theta^i = \langle \theta_1^i, \ldots, \theta_n^i\rangle$. Suppose that $\tau = \sigma(\theta^1, \ldots, \theta^m)$ is an equivalence relation on $A^n$. We need to show $\tau \in \mathcal{M}$. Since $\mathcal{N}$ is closed under primitive positive definitions, $\tau \in \mathcal{N}$. Hence $\tau = \langle \tau_1, \ldots, \tau_n\rangle$ for some $\tau_1, \ldots, \tau_n \in \mathrm{Con}\mathbf{A}$. Also, $\sigma$ must act coordinate-wise, so that $\tau_i = \sigma(\theta_1^i, \ldots, \theta_m^i)$. The assignment $\langle \theta_1^i, \ldots, \theta_m^i\rangle \mapsto \tau_i$ must be order preserving since it comes from a primitive positive definition. Since **L** is OPC and $\mathbf{L} \cong \mathrm{Con}\mathbf{A}$, Con**A** is also OPC. Therefore, there is an $m$-ary polynomial $p$ on Con**A** which interpolates the assignment – that is $p(\theta_1^i, \ldots, \theta_m^i) = \tau_i$ for each $i$. Thus there is a lattice term $t$ and elements $c_1, \ldots, c_k \in \mathrm{Con}\mathbf{A}$ so that $t^{\mathrm{Con}\mathbf{A}}(\theta_1^i, \ldots, \theta_m^i, c_1, \ldots, c_k) = \tau_i$ for each $i$. For $i = 1, \ldots, k$, let $\bar{c}_i = \langle c_i, c_i, \ldots, c_i\rangle$ ($n$ coordinates). Then $t^{\mathrm{Eq}(A^n)}(\theta^1, \ldots, \theta^m, \bar{c}_1, \ldots, \bar{c}_k) = \tau$. Since $\mathcal{M}$ contains the diagonal, each $\bar{c}_i$ is in $\mathcal{M}$. Since $\mathcal{M}$ also contains each $\theta^i$ and is a sublattice of $\mathrm{Eq}(A^n)$, it follows that $\tau \in \mathcal{M}$ as desired. $\mathcal{M}$ is closed under the appropriate primitive positive definitions and, hence, is the congruence lattice of an algebra on $A^n$. Since $\mathcal{M} \cong \mathbf{M}$, we have that **M** is representable. □

Since every subdirect power of **2** contains the diagonal, and since every finite distributive lattice is a subdirect power of **2**, this theorem gives as a corollary the well known fact that every finite distributive lattice is representable.

## 6. The future

We believe there is potential for extending the methods we have begun to explore here. We want to mention some possibilities in this section.

To begin with, a glaring omission from the list of constructions we have formed is homomorphisms. It should be useful to know that the class of representable finite lattices is closed under homomorphisms. The techniques of tame congruence theory might lend themselves to this pursuit. Suppose **A** is a finite algebra and $p$ is an idempotent unary polynomial of **A**. Tame congruence theory provides an algebra on the image of $p$ whose congruence lattice is a homomorphic image of Con**A**. For a discussion of tame congruence theory, see [2].

Theorem 5.1 shows that the study of which lattices of equivalence relations on a finite set $A$ are congruence lattices of algebras on $A$ can be reduced to the study of the subalgebras of an algebra $\langle \mathrm{Eq}(A), P\rangle$ where the operations in $P$ are certain polynomials of the equivalence relation lattice. This observation would be more useful if we could describe the polynomials included in $P$. OPC lattices work well in Theorem 5.2 because in an OPC lattice, polynomials can be found to interpolate any order preserving operation. The truth is that we do not need to be able to interpret any order preserving operation – only those arising from primitive positive formulas. Thus a study of the operations arising from primitive positive formulas and interpolation on finite lattices should be able to extend Theorem 5.2 to larger classes of finite lattices.

A corollary of Lemma 3.5 is that the class of non-representable finite lattices is closed under products. It might be useful to find operations under which this class is closed. For example, a proof that the class of non-representable finite lattices is closed under subintervals would show that the class is empty (since **1** and **2** are subintervals of any nontrivial lattice).

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