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Reflexive relations on algebras with Boolean lattice reducts

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Abstract. For any algebra $\mathbf{A}$, let $\mathbf{Ref}(\mathbf{A})$ be the algebra of compatible reflexive binary relations on $\mathbf{A}$ under intersection, composition, and converse with the universal and identity relations as constants. We characterize all $\mathbf{Ref}(\mathbf{A})$ where $\mathbf{A}$ is a finite algebra with a Boolean lattice reduct.

1. Introduction

For any algebra $\mathbf{A}$, let $\mathrm{Ref}(\mathbf{A})$ be the set of all compatible reflexive binary relations on $\mathbf{A}$. Let $\mathbf{Ref}(\mathbf{A}) = \langle \mathrm{Ref}(\mathbf{A}), \cap, \circ, \cdot^{\cup}, 1_A, \delta_A \rangle$ be the algebra of compatible reflexive binary relations on $\mathbf{A}$ under the operations of intersection, composition, and converse with the universal relation (1_A) and identity relation (δ_A) as constants. In this paper we give a characterization of $\mathbf{Ref}(\mathbf{A})$ when $\mathbf{A}$ is a finite algebra with a Boolean lattice reduct.

Let $\mathbf{A}$ be a finite algebra with a Boolean lattice reduct. First, we follow results given by Gena Boerker in her Masters thesis [1] to realize $\mathbf{Ref}(\mathbf{A})$ as a subalgebra of a direct power of $\mathbf{Ref}(\mathbf{2})$, where $\mathbf{2}$ is the two-element lattice. We then employ a result from [4] to conclude that every subalgebra of a direct power of $\mathbf{Ref}(\mathbf{2})$ is the algebra of compatible reflexive binary relations of some finite algebra with a Boolean lattice reduct. Thus, a finite algebra $\mathbf{R}$ is isomorphic to $\mathbf{Ref}(\mathbf{A})$ for a finite algebra $\mathbf{A}$ with a Boolean lattice reduct if and only if $\mathbf{R}$ is in $\mathrm{SP}(\mathbf{Ref}(\mathbf{2})) = \mathrm{HSP}(\mathbf{Ref}(\mathbf{2}))$.

A quick look at $\mathbf{Ref}(\mathbf{2})$ will then show that an algebra $\langle R, \cap, \circ, \cdot^{\cup}, 1, 0 \rangle$ is in $\mathrm{HSP}(\mathbf{Ref}(\mathbf{2}))$ if and only if $\langle R, \cap, \circ, 1, 0 \rangle$ is a bounded distributive lattice for which $\cdot^{\cup}$ is an involution (an automorphism which is its own inverse).

2. Decomposing algebras of reflexive relations

In this section, we employ standard tools used in tame congruence theory to find decompositions of algebras of the form $\mathbf{Ref}(\mathbf{A})$. Suppose that p is an idempotent unary polynomial of an algebra $\mathbf{A}$. Let $U = p(\mathbf{A})$. By the algebra on U induced by $\mathbf{A}$, we mean the algebra with universe U whose operations are all polynomials of $\mathbf{A}$ under which U is closed. We will denote this algebra

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by $\mathbf{A}|_U$. For any relation $R \in \mathrm{Ref}(\mathbf{A})$, let $p(R) = \{\langle p(a), p(b)\rangle : \langle a, b\rangle \in R\}$. It is easy to prove (using the idempotence of p) that $p(R) = R \cap (U \times U)$. The proof of Lemma 2.3 of [2] can be easily modified to prove the following lemma. This lemma is a special case of Theorem 2.3 of [3], which says that in this environment p actually induces a *relational clone homomorphism* from the relations compatible with $\mathbf{A}|_A$ to the relations compatible with $\mathbf{A}|_U$.

Lemma 2.1. *Suppose that p is an idempotent unary polynomial of an algebra $\mathbf{A}$. Then the map $R \mapsto p(R)$ is a surjective homomorphism from $\mathbf{Ref}(\mathbf{A})$ onto $\mathbf{Ref}(\mathbf{A}|_U)$.*

We will employ this lemma with specific lattice polynomials. If $a < b$ in a lattice $\mathbf{L}$, then let $p_{ab}(x) = (x \vee a) \wedge b$. Note that p_{ab} is an idempotent unary polynomial of $\mathbf{L}$ whose range is the interval $[a, b]$. We first need to know that these polynomials are adequate to separate reflexive relations in Boolean lattices. The next lemma follows from Theorem 46 of [1]—which was the inspiration of this paper.

Lemma 2.2. *Suppose that $\mathbf{B}$ is a finite algebra with a Boolean lattice reduct. Suppose that $R \subset S$ in* $\mathrm{Ref}(\mathbf{B})$*. There exists a coving pair $u \prec v$ so that $p_{uv}(R) \neq p_{uv}(S)$.*

Proof. Since $R \subset S$, there is some pair $\langle x, y\rangle \in S - R$. Since $\langle x, y\rangle \in S$, then we also have $\langle x, x \wedge y\rangle = \langle x, y\rangle \wedge \langle x, x\rangle \in S$ and $\langle x \wedge y, y\rangle = \langle x, y\rangle \wedge \langle y, y\rangle \in S$. Since $\langle x, x \wedge y\rangle \vee \langle x \wedge y, y\rangle = \langle x, y\rangle$, one of these ordered pairs is not in R. Therefore, S contains a comparable pair which is not in R. Without loss of generality, suppose that $a < b$ and that $\langle a, b\rangle \in S - R$. Let $a_1, \ldots, a_n$ be the covers of a in the interval $[a, b]$ (note that finiteness is used here). Since $\langle a, b\rangle \in S$, then we know $\langle a, a_i\rangle = \langle a, b\rangle \wedge \langle a_i, a_i\rangle \in S$ for each i. Since $[a, b]$ must be a Boolean lattice, $b = a_1 \vee a_2 \vee \cdots \vee a_n$. Therefore, if $\langle a, a_i\rangle$ were in R for every i, then the join of these pairs—which is $\langle a, b\rangle$—would be in R. Since this is not the case, there is some i for which $\langle a, a_i\rangle \in S - R$. Let $u = a$, $v = a_i$, and $U = p_{uv}(\mathbf{B})$. Then $\langle u, v\rangle \in S \cap (U \times U) = p_{uv}(S)$, but $\langle u, v\rangle \notin R \cap (U \times U) = p_{uv}(R)$. Thus $p_{uv}(R) \neq p_{uv}(S)$ as desired. □

Suppose that $\mathbf{B}$ is a finite algebra with a Boolean lattice reduct and that $u \prec v$ in $\mathbf{B}$. Let $U_{uv} = p_{uv}(\mathbf{B}) = \{u, v\}$. Then U_{uv} is closed under the lattice operations of $\mathbf{B}$, so $\mathbf{B}|_{U_{uv}}$ has a two-element lattice as a reduct. This means that $\mathbf{B}|_{U_{uv}}$ is isomorphic to an algebra on $\{0, 1\}$ with $\mathbf{2}$ as a reduct and that $\mathbf{Ref}(\mathbf{B}|_{U_{uv}})$ can be embedded into $\mathbf{Ref}(\mathbf{2})$. By Lemmas 2.1 and 2.2, we know that $\mathbf{Ref}(\mathbf{B})$ can be embedded into $\prod\{\mathbf{Ref}(\mathbf{B}|_{U_{uv}}) : u \prec v \in \mathbf{B}\}$. By what we have just said, this product can be embedded in $\prod\{\mathbf{Ref}(\mathbf{2}) : u \prec v \in \mathbf{B}\}$. We have proven the following corollary.

Corollary 2.3. *Suppose that $\mathbf{B}$ is a finite algebra with a Boolean lattice reduct. Then $\mathbf{Ref}(\mathbf{B}) \in \mathrm{SP}_{\mathrm{fin}}(\mathbf{Ref}(\mathbf{2}))$.*

The converse of this corollary follows quickly from the following lemma. In the lemma, $\mathcal{R}_2(\mathbf{A})$ is all compatible binary relations on an algebra $\mathbf{A}$.

Lemma 2.4. (Corollary 3.9 of [4]) *Suppose that $\mathcal{R}$ is a system of binary relations on a finite set A which is closed under composition, converse, and intersection and which contains A^2 and the binary diagonal δ_A. If the relations in $\mathcal{R}$ are compatible with a majority operation on A, then there is an algebra $\mathbf{A}$ on A with $\mathcal{R} = \mathcal{R}_2(\mathbf{A})$.*

Suppose that $\mathbf{R} \in \mathrm{SP}_{\mathrm{fin}}(\mathbf{Ref}(\mathbf{2}))$. Then we can realize $\mathbf{R}$ as a subalgebra of $\mathbf{Ref}(\mathbf{2}^n)$ for some finite n. The relations in $\mathbf{R}$ must be compatible with the lattice operations of $\mathbf{2}^n$, so they are compatible with the majority operation of $\mathbf{2}^n$. By the previous lemma, there is an algebra $\mathbf{A}$ on $\{0,1\}^n$ with $\mathcal{R}_2(\mathbf{A}) = \mathbf{R}$. Since the relations in $\mathbf{R}$ are all reflexive (by assumption), this means that $\mathbf{Ref}(\mathbf{A}) = \mathbf{R}$. Since the relations in $\mathbf{R}$ are compatible with the lattice operations of $\mathbf{2}^n$ (a Boolean lattice), we can assume that these operations are among the operations of $\mathbf{A}$. This along with Corollary 2.3 proves the next lemma.

Lemma 2.5. *A finite algebra $\mathbf{R}$ is isomorphic to $\mathbf{Ref}(\mathbf{B})$ for a finite algebra $\mathbf{B}$ with a Boolean lattice reduct if and only if $\mathbf{R} \in \mathrm{SP}_{\mathrm{fin}}(\mathbf{Ref}(\mathbf{2}))$.*

We now observe that $\mathrm{HSP}(\mathbf{Ref}(\mathbf{2})) = \mathrm{SP}(\mathbf{Ref}(\mathbf{2}))$. The algebra $\mathbf{Ref}(\mathbf{2})$ has four elements: δ_2, 1_2, $\leq$ and $\geq$. It is easy to check that $\langle \mathrm{Ref}(\mathbf{2}), \cap, \circ, 1_2, \delta_2 \rangle$ is a bounded distributive lattice (with $\circ$ as join) and that converse is an involution of this lattice. It follows that $\mathbf{Ref}(\mathbf{2})$ has a majority operation and generates a congruence distributive variety. By Jonsson's Lemma, every subdirectly irreducible member of $\mathrm{HSP}(\mathbf{Ref}(\mathbf{2}))$ is in $\mathrm{HS}(\mathbf{Ref}(\mathbf{2}))$. However, $\mathbf{Ref}(\mathbf{2})$ is simple and has only one nontrivial proper subalgebra (the one containing only 1_2 and δ_2), which is also simple. Therefore, the only subdirectly irreducible algebras in $\mathrm{HSP}(\mathbf{Ref}(\mathbf{2}))$ are $\mathbf{Ref}(\mathbf{2})$ and its two-element subalgebra, so $\mathrm{HSP}(\mathbf{Ref}(\mathbf{2})) = \mathrm{SP}(\mathbf{Ref}(\mathbf{2}))$. Lemma 2.5 can now be restated as

Lemma 2.6. *A finite algebra $\mathbf{R}$ is isomorphic to $\mathbf{Ref}(\mathbf{B})$ for a finite algebra $\mathbf{B}$ with a Boolean lattice reduct if and only if $\mathbf{R}$ is in $\mathrm{HSP}(\mathbf{Ref}(\mathbf{2})) = \mathrm{SP}(\mathbf{Ref}(\mathbf{2}))$.*

3. Equations for $\mathrm{HSP}(\mathbf{Ref}(\mathbf{2}))$

We now find equations defining $\mathrm{HSP}(\mathbf{Ref}(\mathbf{2}))$. As we said above, the reduct $\langle \mathrm{Ref}(\mathbf{2}), \cap, \circ, 1_2, \delta_2 \rangle$ is a bounded distributive lattice, and converse is an involution of this lattice. These properties are equational, so they extend to every algebra in $\mathrm{HSP}(\mathbf{Ref}(\mathbf{2}))$. Conveniently, these equations are enough.

Lemma 3.1. *Suppose that $\mathbf{L}$ is a bounded distributive lattice and $f\colon L \to L$ is an involution. Then $\langle L, \wedge, \vee, f, 1, 0 \rangle \in \mathrm{HSP}(\mathbf{Ref}(\mathbf{2}))$.*

Proof. Let $\mathbf{L}' = \langle L, \wedge, \vee, f, 1, 0\rangle$. We need only show that for every $a \neq b \in \mathbf{L}'$, there is a homomorphism from $\mathbf{L}'$ to $\mathbf{Ref}(\mathbf{2})$ that does not identify a and b. It will then follow that $\mathbf{L}'$ can be embedded into a direct power of $\mathbf{Ref}(\mathbf{2})$. Let $a \neq b \in \mathbf{L}'$. Since $\mathbf{L}$ is a distributive lattice, there is a congruence α of $\mathbf{L}$ so that $\langle a, b\rangle \notin \alpha$ and so that $|\mathbf{L}/\alpha| = 2$. Let $\beta = \alpha \cap f(\alpha)$. Since f is an automorphism of $\mathbf{L}$, $f(\alpha)$ is a congruence of $\mathbf{L}$, and so is β. But then β is also a congruence of $\mathbf{L}'$ because

$$f(\beta) = f(\alpha \cap f(\alpha)) = f(\alpha) \cap f(f(\alpha)) = f(\alpha) \cap \alpha = \beta$$

(where the second equality follows from the bijectivity of f and the third from the fact that f is an involution). Thus β is a congruence of $\mathbf{L}'$ that separates a and b.

It may be that every element of $\mathbf{L}'$ is β-congruent to either 1 or 0, so that β has precisely two classes. In this case, f acts as the identity on $\mathbf{L}'/\beta$, and $\mathbf{L}'/\beta$ is isomorphic to the two-element subalgebra of $\mathbf{Ref}(\mathbf{2})$. In this case, we have a homomorphism from $\mathbf{L}'$ to $\mathbf{Ref}(\mathbf{2})$ separating a and b.

Suppose then that there is some $x \in \mathbf{L}'$ that is not β-equivalent to 1 or to 0. Now α has exactly two classes: $1/\alpha$ and $0/\alpha$. The same is true of $f(\alpha)$. This means that β has at most four classes: $(1/\alpha) \cap (1/f(\alpha))$, $(1/\alpha) \cap (0/f(\alpha))$, $(0/\alpha) \cap (1/f(\alpha))$, and $(0/\alpha) \cap (0/f(\alpha))$. Because $1/\alpha$ and $0/\alpha$ are disjoint, and because $1/f(\alpha)$ and $0/f(\alpha)$ are disjoint, these four possible classes are pairwise disjoint. Also, $(1/\alpha) \cap (1/f(\alpha))$ and $(0/\alpha) \cap (0/f(\alpha))$ are not empty. Moreover, since we are assuming the existence of an element not β-related to 1 or to 0, at least one of the other classes is not empty. But observe that

$$\begin{aligned} f((0/\alpha) \cap (1/f(\alpha))) &= f(0/\alpha) \cap f(1/f(\alpha)) \\ &= f(0)/f(\alpha) \cap f(1)/f(f(\alpha)) \\ &= (0/f(\alpha)) \cap (1/\alpha). \end{aligned}$$

(The first equality follows from the bijectivity of f. The second and third equalities are true because f is an involution of $\mathbf{L}$.) This means that $(0/\alpha) \cap (1/f(\alpha))$ is nonempty if and only if $(1/\alpha) \cap (0/f(\alpha))$ is nonempty. Since we know at least one is not empty, neither of them is empty. Thus $\mathbf{L}/\beta$ is a four-element lattice with a non-identity involution (induced by f). The only possibility is that $\mathbf{L}/\beta$ is isomorphic to $\mathbf{2}^2$ and that f is the automorphism that exchanges the atoms. This means that $\mathbf{L}'/\beta \cong \mathbf{Ref}(\mathbf{2})$, so that we have a homomorphism from $\mathbf{L}'$ onto $\mathbf{Ref}(\mathbf{2})$ that separates a and b.

We have shown that for all $a \neq b \in \mathbf{L}'$ there is a homomorphism from $\mathbf{L}'$ to $\mathbf{Ref}(\mathbf{2})$ that separates a and b. It follows that $\mathbf{L}'$ can be embedded into a direct power of $\mathbf{Ref}(\mathbf{2})$. □

Combining Lemma 2.6 with this lemma we have proven our main theorem.

Theorem 3.2. *Suppose that* $\mathbf{R} = \langle R, \cap, \circ, \cdot^{\cup}, 1, \delta\rangle$ *is a finite algebra of the same type as algebras of compatible reflexive relations. The following are equivalent:*

(1) $\mathbf{R} \cong \mathbf{Ref}(\mathbf{B})$ *for some finite algebra* $\mathbf{B}$ *with a Boolean lattice reduct.*
(2) $\langle R, \cap, \circ, 1, \delta \rangle$ *is a bounded distributive lattice and* $\cdot^{\cup}$ *is an involution of this lattice.*
(3) $\mathbf{R} \in \mathrm{HSP}(\mathbf{Ref}(\mathbf{2})) = \mathrm{SP}(\mathbf{Ref}(\mathbf{2}))$.

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