Lattices of Equivalence Relations Closed Under the Operations of Relation Algebras

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ABSTRACT. One of the longstanding problems in universal algebra is the question of which finite lattices are isomorphic to the congruence lattices of finite algebras. This question can be phrased as which finite lattices can be represented as lattices of equivalence relations on finite sets closed under certain first-order formulas. We generalize this question to a different collection of first-order formulas, giving examples to demonstrate that our new question is distinct. We then note that every lattice $\mathbf{M}_n$ can be represented in this new way.

1. Introduction

One of the longstanding problems in universal algebra is

Problem 1.1. Finite Congruence Lattice Representation Problem (FCLRP): *For which finite lattices* $\mathbf{L}$ *is there a finite algebra* $\mathbf{A}$ *with* $\mathbf{L} \cong \mathrm{Con}\mathbf{A}$*?*

A *primitive positive formula* is a first-order formula of the form $\exists \wedge$ (atomic). Suppose that $\mathcal{R}$ is a set of relations on a finite set A. Let $\mathrm{PPF}(\mathcal{R})$ be the set of all relations on A definable using primitive positive formulas and relations from $\mathcal{R}$. Let $\mathrm{Eq}(\mathcal{R})$ be the set of all equivalence relations in $\mathcal{R}$. It follows from [2, 7] that $\mathcal{R}$ is the set of all universes of direct powers of an algebra $\mathbf{A}$ with universe A if and only if $\mathrm{PPF}(\mathcal{R}) = \mathcal{R}$. (For references on similar characterizations, the reader can see [6].) Therefore, Problem 1.1 can be restated in the following way.

Problem 1.2. *For which finite lattices* $\mathbf{L}$ *is there a lattice* $\mathcal{L}$ *of equivalence relations on a finite set so that* $\mathbf{L} \cong \mathcal{L}$ *and* $\mathcal{L} = \mathrm{Eq}(\mathrm{PPF}(\mathcal{L}))$*?*

A natural extension of this problem is to consider first-order definitions employing types of formulas other than primitive positive formulas. We suggest replacing primitive positive formulas with any first-order formulas using at most three variables. If $\mathcal{R}$ is a set of relations on a finite set A, let $\mathrm{FO3}(\mathcal{R})$ be the set of all relations on A definable using first-order formulas with at most three variables and relations from $\mathcal{R}$. Our extension of 1.2 can be stated as:

Problem 1.3. *For which finite lattices* $\mathbf{L}$ *is there a lattice* $\mathcal{L}$ *of equivalence relations on a finite set so that* $\mathbf{L} \cong \mathcal{L}$ *and* $\mathcal{L} = \mathrm{Eq}(\mathrm{FO3}(\mathcal{L}))$*?*

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Our interest in first-order formulas with three variables stems from a connection with relation algebras. A *relation algebra* is an algebra $\mathbf{A} = \langle A, +, \bar{\ }, ;, \cdot^{\cup}, 1' \rangle$ with operations intended mimic the operations of union, complement, composition, converse, and identity on binary relations. A relation algebra $\mathbf{A}$ is *representable* if there is a set of binary relations $\mathcal{R}$ on a set B so that $\mathbf{A}$ is isomorphic to the algebra $\langle \mathcal{R}, \cup, \bar{\ }, \circ, \cdot^{\cup}, 1'_B \rangle$. A set $\mathcal{R}$ of binary relations on a finite set A is closed under the relation algebra operations if and only if every binary relation on A definable with a first-order formula with at most three variables and relations in $\mathcal{R}$ is already in $\mathcal{R}$ (see Theorem 3.32 of [3] or page 172 of [8]). For any set $\mathcal{R}$ of binary relations on a set A, let $\mathrm{RA}(\mathcal{R})$ be the relation algebra generated by $\mathcal{R}$. Then the above problem becomes:

Problem 1.4. *For which finite lattices* $\mathbf{L}$ *is there a lattice* $\mathcal{L}$ *of equivalence relations on a finite set so that* $\mathbf{L} \cong \mathcal{L}$ *and* $\mathcal{L} = \mathrm{Eq}(\mathrm{RA}(\mathcal{L}))$*?*

We wonder whether this problem may prove to be more tractable than FCLRP. For instance, it is not known for which $n < \omega$ $\mathbf{M}_n$ is representable (in the usual sense); however, we show below that $\mathbf{M}_n$ is representable in our new sense for all $n < \omega$. Our problem is a natural generalization of the FCLRP, requiring closure under a broader type of first-order formula. Since it relates naturally to relation algebras, and since relation algebras give an immediate contribution in our examples, the authors hope that this problem will perhaps attract the attention of other relation-algebraists, who will apply their tools to the FCLRP and related problems.

2. Examples

In this section we give two examples $\mathcal{L}$ and $\mathcal{M}$ of lattices of equivalence relations on finite sets. In the first example, $\mathrm{Eq}(\mathrm{PPF}(\mathcal{L})) = \mathcal{L}$ but $\mathrm{Eq}(\mathrm{RA}(\mathcal{L})) \neq \mathcal{L}$. In the second example, $\mathrm{Eq}(\mathrm{RA}(\mathcal{M})) = \mathcal{M}$ but $\mathrm{Eq}(\mathrm{PPF}(\mathcal{M})) \neq \mathcal{M}$. This demonstrates that these two notions are indeed distinct.

First, let $\mathbf{2}$ be the two-element lattice with universe $\{0, 1\}$. Let $\mathbf{A} = \mathbf{2}^2$, and let $\mathcal{L} = \mathrm{Con}\mathbf{A}$. Then $\mathcal{L}$ contains four equivalence relations – the identity relation, the universal relation, and the kernels of the projection homomorphisms. The projection kernels are the relations η_0 and η_1 defined so that $(x_0, x_1)\, \eta_0\, (y_0, y_1)$ when $x_0 = y_0$ and $(x_0, x_1)\, \eta_1\, (y_0, y_1)$ when $x_1 = y_1$. Since $\mathcal{L}$ is a congruence lattice, $\mathrm{Eq}(\mathrm{PPF}(\mathcal{L})) = \mathcal{L}$. However, $\mathrm{RA}(\mathcal{L})$ also contains the equivalence relation

$$\gamma = 1' \cup \overline{(\eta_0 \cup \eta_1)}$$

which is not in $\mathcal{L}$, so $\mathrm{Eq}(\mathrm{RA}(\mathcal{L})) \neq \mathcal{L}$. Note that the relation γ can also be defined with this first-order formula which only uses *two* variables:

$$x\, \gamma\, y \leftrightarrow (x = y) \vee \neg[(x\, \eta_0\, y) \vee (x\, \eta_1\, y)].$$

Thus $\mathcal{L}$ is closed under primitive positive definitions but not under the operations of relation algebras or first-order definitions using at most three variables.

For our second example, suppose that $p \geq 5$ is prime. We consider $\mathrm{Con}(\mathbb{Z}_p^2)$, which is a copy of $\mathbf{M}_{p+1}$ consisting of the identity 1', the universal relation 1, and

$p+1$ atoms $\eta_0, \eta_1, \alpha_1, \ldots, \alpha_{p-1}$, given by

$$\begin{aligned}
\langle x_0, x_1\rangle \,\eta_0\, \langle y_0, y_1\rangle &\leftrightarrow x_0 = y_0\\
\langle x_0, x_1\rangle \,\eta_1\, \langle y_0, y_1\rangle &\leftrightarrow x_1 = y_1\\
\langle x_0, x_1\rangle \,\alpha_1\, \langle y_0, y_1\rangle &\leftrightarrow 1x_0 - x_1 = 1y_0 - y_1\\
\langle x_0, x_1\rangle \,\alpha_2\, \langle y_0, y_1\rangle &\leftrightarrow 2x_0 - x_1 = 2y_0 - y_1\\
&\vdots\\
\langle x_0, x_1\rangle \,\alpha_k\, \langle y_0, y_1\rangle &\leftrightarrow kx_0 - x_1 = ky_0 - y_1\\
&\vdots\\
\langle x_0, x_1\rangle \,\alpha_{p-1}\, \langle y_0, y_1\rangle &\leftrightarrow (p-1)x_0 - x_1 = (p-1)y_0 - y_1
\end{aligned}$$

Suppose that $1 \le n < p-2$ and let $\mathcal{M} = \{1, 1', \eta_0, \eta_1, \alpha_1, \ldots, \alpha_n\}$. It follows from [5] that $\mathrm{Eq}(\mathrm{RA}(\mathcal{M})) = \mathcal{M}$. (This result is not explicitly stated in the paper, although it can be extracted from it; readers who wish to see a bottom-up proof should see the extended version of the current paper [1], Lemma 2.1.) However, the relation α_{p-1} (which is not in $\mathcal{M}$) can be defined from η_0, η_1, and α_1 with a primitive positive formula by

$$a\,\alpha_{p-1}\,b \leftrightarrow \exists c, d\,[(a\,\eta_0\,c) \wedge (c\,\eta_1\,b) \wedge (a\,\eta_1\,d) \wedge (d\,\eta_0\,b) \wedge (c\,\alpha_1\,d)]\,.$$

Thus $\mathrm{Eq}(\mathrm{PPF}(\mathcal{M})) \neq \mathcal{M}$. The lattice $\mathcal{M}$ is closed under the operations of relation algebras and first-order definitions using at most three variables but not under primitive positive definitions.

This second example has the following interesting consequence. If $n \ge 1$ and if $p \ge 5$ is a prime greater than $n+2$, then the lattice $\mathcal{M}$ in the example gives a lattice of equivalence relations closed under the operations of relation algebras which is isomorphic to $\mathbf{M}_{n+2}$. Note that $\mathbf{M}_1$ and $\mathbf{M}_2$ can easily be represented by letting $\mathcal{M}$ be $\{1, 1', \eta_0\}$ and $\{1, 1', \eta_0, \eta_1\}$, respectively. Thus we have

Theorem 2.1. *For any positive integer n, there is a lattice $\mathcal{M}$ of equivalence relations on a finite set so that $\mathcal{M} \cong \mathbf{M}_n$ and $\mathrm{Eq}(\mathrm{RA}(\mathcal{M})) = \mathrm{Eq}(\mathrm{FO3}(\mathcal{M})) = \mathcal{M}$.*

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